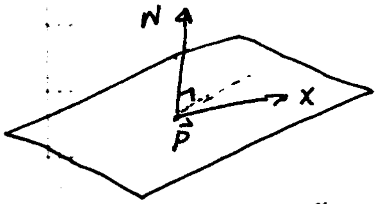
Math 32 9/9/09

Planes (in $\mathbb{R}^3$)



A plane is completely determined by choosing a point $\vec{P}$ that it passes through and a "normal direction" $\vec{N}$ which is perpindicular (or normal) to the plane.

$\vec{X}$ is in the plane iff

$$(\vec{X}-\vec{P})\cdot\vec{N} = 0.$$

(i.e. $(\vec{X}-\vec{P}) \perp \vec{N}$).

e.g. The plane through $\vec{P}=(1,2,1)$ perpendicular to $\vec{N}=(-2,3,7)$ has the equation

($\vec{X} = (x,y,z)$)

$$(x-1,\ y-2,\ z-1)\cdot(-2,3,7) = 0.$$

or $-2(x-1) + 3(y-2) + 7(z-1) = 0.$

In general, if $\vec{P} = (x_0, y_0, z_0)$
$\vec{N} = (a, b, c)$

the plane has the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

This can also be written in the form

$$ax + by + cz = k \qquad \left[\vec{X} \cdot \vec{N} = \vec{N} \cdot \vec{P} \right].$$

where k is a constant. $(\vec{N} \cdot \vec{P})$.

e.g. Find a point in the plane.

$$3x - 2y + 7z = 14$$

answer: (there are many).

$$(0, 0, 2).$$

$$(1, 1, 13/7).$$

e.g. $Q = (2, 1, 3) \qquad P = (-1, -1, -1)$

$$N = (3, 1, 1).$$

Find the intersection point of the line through $\vec{P}$ with direction $\vec{N}$ and the plane through Q with normal $\vec{N}$.

The line is given by.

$$\vec{X} = \vec{P} + t\vec{N} \qquad t \in \mathbb{R}$$

and the plane by.

$$(\vec{X} - \vec{Q}) \cdot \vec{N} = 0$$

The point of intersection occurs when t satisfies

$$(\vec{P} + t\vec{N} - \vec{Q}) \cdot \vec{N} = 0.$$

or $(\vec{P} - \vec{Q}) \cdot \vec{N} = -t \|\vec{N}\|^2$.

$$t = \frac{(\vec{Q} - \vec{P}) \cdot \vec{N}}{\|\vec{N}\|^2}$$

for our example this is.

$$t = \frac{(3,2,4) \cdot (3,1,1)}{11} = \frac{15}{11}.$$

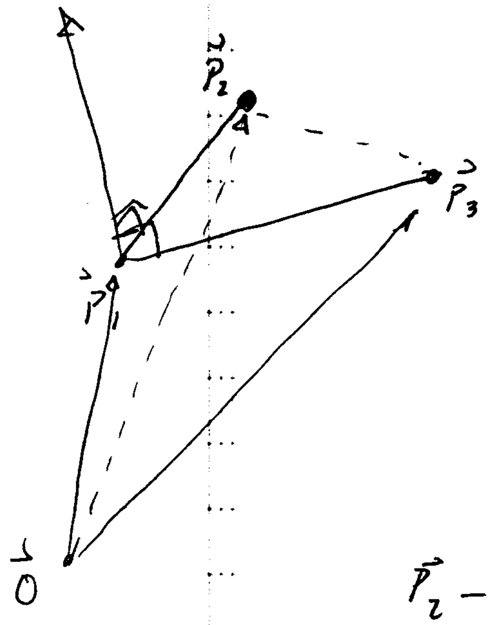
So the point of intersection is

$$(-1,-1,-1) + \frac{15}{11}(3,1,1) = \left(\frac{34}{11}, \frac{4}{11}, \frac{4}{11}\right).$$

e.g. Find the equation of the plane passing through

$P_1 = (1, 1, 4)$, $P_2 = (-1, -2, -3)$

$P_3 = (10, 2, 0)$



We need (e.g.) a vector perpindicular to both

~~$\vec{P_1P_2}$~~ and ~~$\vec{P_1P_3}$~~

$\vec{P_2} - \vec{P_1}$ and $\vec{P_3} - \vec{P_1}$

$\vec{P_2} - \vec{P_1} = (-2, -3, -7)$

$\vec{P_3} - \vec{P_1} = (9, 1, -4)$.

With $\vec{N} = (a, b, c)$ we need both

$(-2, -3, -7) \cdot (a, b, c) = 0$

and $(9, 1, -4) \cdot (a, b, c) = 0$.

$-2a - 3b - 7c = 0$.

$9a + b - 4c = 0$.

multiply 2nd equation by 3 and add to 1st.

$$-2a - 3b - 7c = 0.$$

$$27a + 3b - 12c = 0.$$

$$25a - 19c = 0.$$

with $a = 19$, $c = 25$ we get

$$2 \cdot 19 + 3b + 7 \cdot 25 = 0.$$

$$3b = -175 - 38 = -213$$

$$b = -\frac{213}{3.} = -71$$

$$\vec{W} = (19, -71, 25)$$

An equation for the plane is

$$19(x-1) - 71(y-1) + 25(z-4) = 0.$$

It would be nice to have an easier way to find a vector perpendicular to two given vectors.

e.g. $\vec{i} = (1,0,0)$ $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -3 & -7 \\ 9 & 1 & -4 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -3 & -7 \\ 1 & -4 \end{vmatrix} - \vec{j} \begin{vmatrix} -2 & -7 \\ 9 & -4 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & -3 \\ 9 & 1 \end{vmatrix}$$

$$= i \Big((-3)(-4) - (-7)(1) \Big) - \vec{j} \Big((-2)(-4) - (-7)(9) \Big)$$

$$+ \vec{k} \Big((-2)(1) - (-3)(9) \Big)$$

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

$$= 19\vec{i} - 71\vec{j} + 25\vec{k} = (19, -71, 25)$$

Cross Product

Def: Given two vectors

$$\vec{A} = (a_1, a_2, a_3)$$

$$\vec{B} = (b_1, b_2, b_3).$$

let
$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i}\,(a_2 b_3 - a_3 b_2) - \vec{j}\,(a_1 b_3 - a_3 b_1) + \vec{k}\,(a_1 b_2 - a_2 b_1).$$

Then $(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{B} = \vec{0}$

Pf: $(\vec{A} \times \vec{B}) \cdot \vec{A} = a_1 (a_2 b_3 - a_3 b_2) - a_2 (a_1 b_3 - a_3 b_1)$

$+ a_3 (a_1 b_2 - a_2 b_1) = 0.$

$(\vec{A} \times \vec{B}) \cdot \vec{B} = b_1 (a_2 b_3 - a_3 b_2) - b_2 (a_1 b_3 - a_3 b_1) + b_3 (a_1 b_2 - a_2 b_1)$

$= 0.$

e.g. Find an equation for the line of intersection of the two planes.

$$x + 3y - z = 7 \quad \text{and} \quad 2x - y + 2z = 4.$$

A vector in the direction of the line of intersection must be perpendicular to both normal vectors.

$$\begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{vmatrix} = 5\vec{i} - 4\vec{j} - 7\vec{k}$$

Now we just need one point in the intersection

$$2x + 6y - 2z = 14$$
$$2x - y + 2z = 4$$

$$4x + 5y = 18$$

$x = y = 2$ works. For this

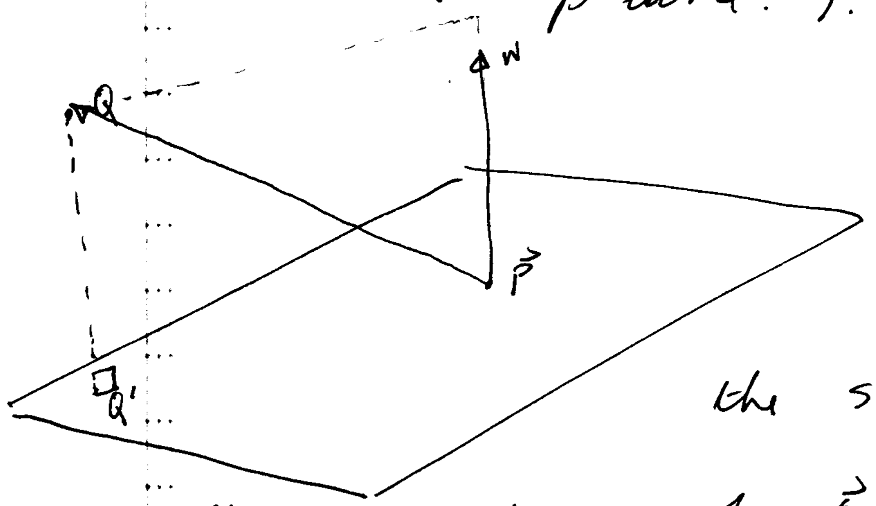
the $2 + 6 - z = 7$ so $z = 1$

So $(2,2,1)$ is a point in the intersection and

$$x = 3 + 5t$$
$$y = 2 - 4t$$
$$z = 1 - 7t$$

gives a parametric equation for the line of intersection

<u>e.g.</u> (Distance between a point and a plane.)



We are looking for the length $\|\vec{Q} - \vec{Q}'\|$. This is the same as the a length of the projection of $\vec{Q} - \vec{P}$ along $\vec{N}$

i.e. $$\|\vec{Q} - \vec{Q}'\| = \left| \frac{(\vec{Q} - \vec{P}) \cdot \vec{N}}{\vec{N} \cdot \vec{N}} \right| \|\vec{N}\|$$

$$= \frac{|(\vec{Q} - \vec{P}) \cdot \vec{N}|}{\|\vec{N}\|}.$$

e.g. Find the distance from

$\vec{Q} = (1, 7, 4)$ to the plane.

$x + 2y + z = 1$.

Pick any point in the plane.

e.g. $(0, 0, 1) = \vec{P}$.

$\vec{Q} - \vec{P} = (1, 7, 3)$.

$\vec{N} = (1, 2, 1)$

$$\text{Distance} = \frac{18}{\sqrt{6}} = 3\sqrt{6}$$

alternatively:

the line $\begin{aligned} x &= 1 + t \\ y &= 7 + 2t \\ z &= 4 + t \end{aligned}$ intersects the plane at.

$$1+t + 2(7+2t) + (4+t) = 1.$$

$$6t = 1 - 1 - 14 - 4 \qquad 6t = -18.$$

$$t = -3.$$

$$(x_0, y_0, z_0) = (-2, 1, 1).$$

Distance from $(-2,1,1)$ to $(1,7,4)$ $= \sqrt{9+36+9} = 3\sqrt{6}$

e.g. Distance between two lines

$$L_1 := \quad \begin{aligned} x &= 1 + t \\ y &= 2 + 2t \\ z &= 3 + 3t \end{aligned}$$

$$L_2 := \quad \begin{aligned} x &= -3 + 2t \\ y &= 2 - t \\ z &= 6 + 5t \end{aligned}$$

The "distance between L_1 and L_2" means

the shortest distance possible between one point of L_1 and some point of L_2.

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -1 & 5 \end{vmatrix} = 13i + \vec{j} - 5\vec{k}$$

is a vector which is perpendicular to both lines.

There are 2 parallel planes with this normal direction containing L_1 and L_2 respectively.

e.g.
The plane containing L_2 is.

$$13(x+3) + (y-2) - 5(z-6) = 0.$$

The distance between this plane and any point on L_1 will give the distance between L_1 and L_2.

Take $\vec{Q} = (1, 2, 3)$ in L_1
$\vec{P} = (-3, 2, 6)$ in L_2.

We need length of the projection of.

$\vec{Q} - \vec{P}$ along $\vec{N} = (13, 1, -5)$.

$$\vec{Q} - \vec{P} = (4, 0, -3)$$

$$\left|\frac{(\vec{Q} - \vec{P}) \cdot \vec{N}}{\|N\|}\right| = \frac{52 + 15}{\sqrt{13^2 + 1 + 5^2}}$$

$$= \frac{77}{\sqrt{195}}.$$