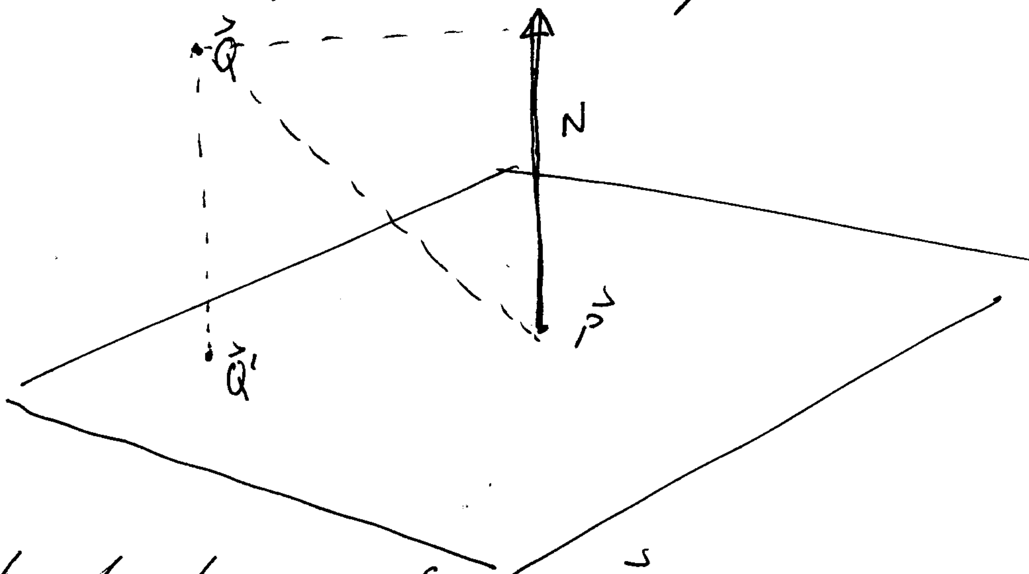
Math 32 9/11/09

Distance between a point and a plane.



The shortest distance from $\vec{Q}$ to the plane through $\vec{P}$ with normal $\vec{N}$ is the length of the projection of $\vec{Q}-\vec{P}$ along $\vec{N}$.

This projection is

$$\frac{(\vec{Q}-\vec{P})\cdot\vec{N}}{\vec{N}\cdot\vec{N}}\vec{N}$$

and its length is

$$\frac{|(\vec{Q}-\vec{P})\cdot\vec{N}|}{\|\vec{N}\|}$$

Distance between two lines

$L_1: \quad \vec{X} = \vec{P_1} + t\vec{A_1}$

$L_2: \quad \vec{X} = \vec{P_2} + t\vec{A_2}$

$\vec{N} = \vec{A_1} \times \vec{A_2}$ is a vector perpindicular to each of the two lines

L_1 is contained in the plane.

$$(\vec{X} - \vec{P_1}) \cdot \vec{N} = 0.$$

L_2 is contained in the plane

$$(\vec{X} - \vec{P_2}) \cdot \vec{N} = 0.$$

Since the planes are parallel, the distance between L_1 and L_2 is the distance between the two planes.

This is the length of the projection of $\vec{P_2} - \vec{P_1}$ along $\vec{N}$.

e.g. Find the distance between the lines

$$L_1: \begin{aligned} x &= 2 + 2t \\ y &= 1 - 5t \\ z &= 3 + t \end{aligned}$$

$$L_2: \begin{aligned} x &= 7 - 4t \\ y &= 4 - 2t \\ z &= -6 + 7t. \end{aligned}$$

We need a vector which is perpendicular to both $(2,-5,1)$ and $(-4,-2,7)$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -5 & 1 \\ -4 & -2 & 7 \end{vmatrix} = \vec{i}\begin{vmatrix} -5 & 1 \\ -2 & 7 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ -4 & 7 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -5 \\ -4 & -2 \end{vmatrix}$$

$$= \vec{i}(-33) - 18\vec{j} - 24\vec{k}$$

let's use

$$\vec{N} = 11\vec{i} + 6j + 8\vec{k}$$

Since L_1 contains $(2,1,3)$

and L_2 contains $(7,4,-6)$

We just need the length of the projection of

$$(2,1,3) - (7,4,-6) = (-5,-3,9)$$

along $(11,6,8)$.

This is

$$\frac{|(-5,-3,9)\cdot(11,6,8)|}{\sqrt{121+36+64}} = \frac{|-55-18+72|}{\sqrt{221}}$$

$$= \frac{1}{\sqrt{221}}$$

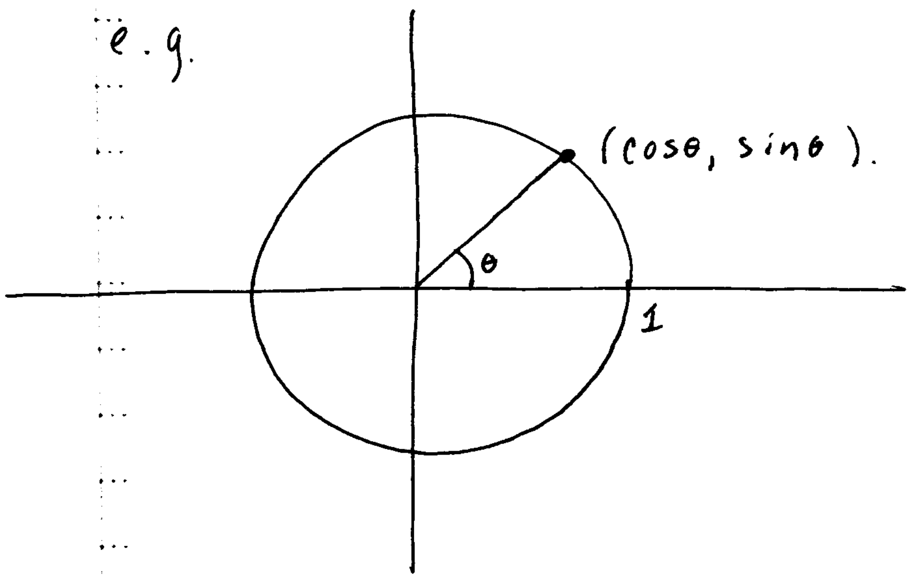
Parametrized Curves

We have already seen an easy example.

$$\vec{X}(t) = \vec{P} + t\vec{A}$$

is a parameterization of a line.

e.g.



As θ runs from 0 to 2π.

The point $(\cos\theta, \sin\theta)$ moves counter clockwise around the circle.

$\vec{X}(t) = (\cos t, \sin t)$ is a parameterization of the circle.

e.g. $\vec{X}(t) = (3\cos 2\pi t, 3\sin 2\pi t)$

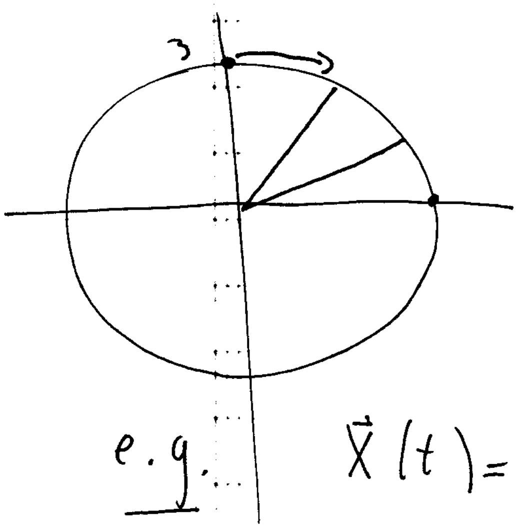
As t runs from 0 to 1.

$\vec{X}(t)$ traces out a circle of radius 3 centered at $\vec{0}$.

$\vec{X}(t)$ again moves counter clockwise.

$\vec{X}(t) = (3\sin 2\pi t, 3\cos 2\pi t)$

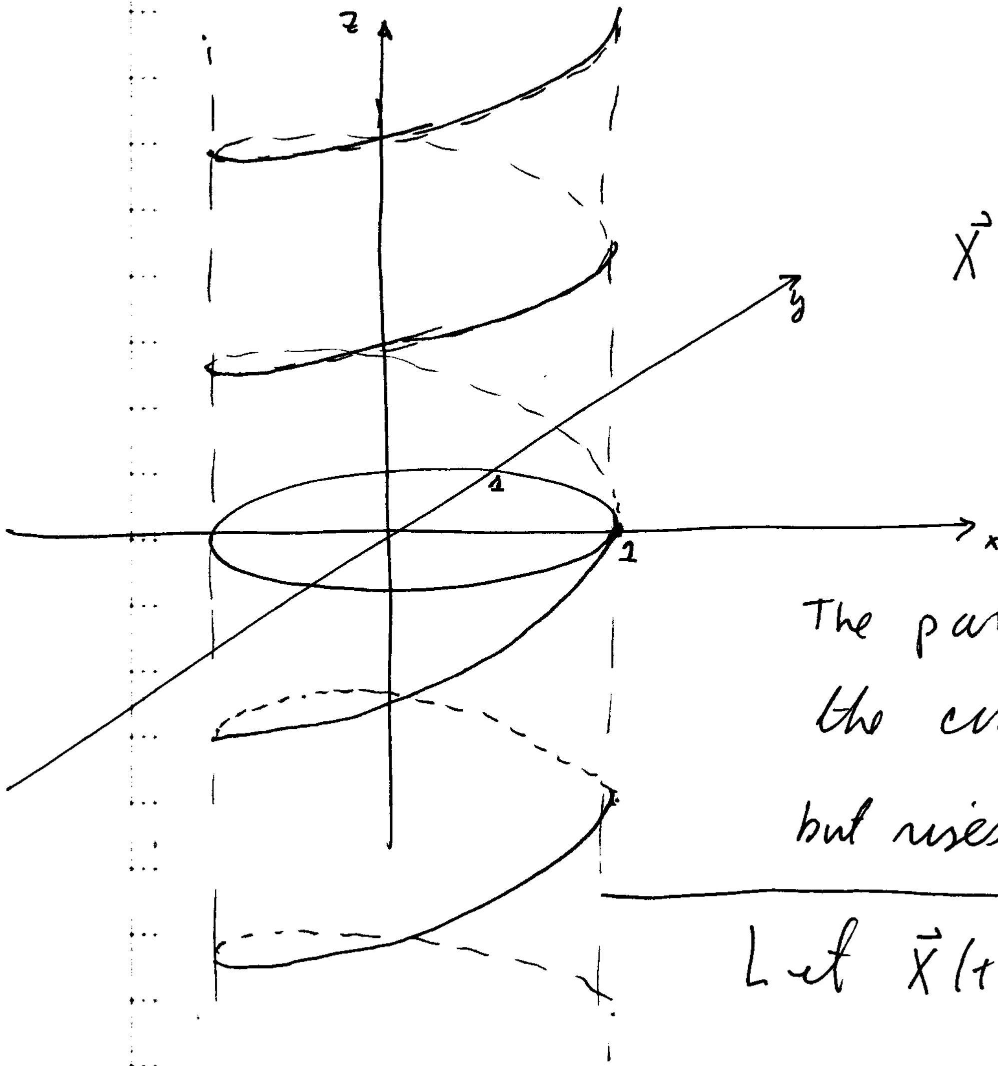
does the same thing, but clockwise.



e.g. $\vec{X}(t) = (3\sin \frac{\pi}{6} t, 3\cos\frac{\pi}{6} t)$.

traces out an angle of $\pi/6$ as t runs from 0 to 1.

e.g. $\vec{X}(t) = \langle \cos t, \sin t, t \rangle$.



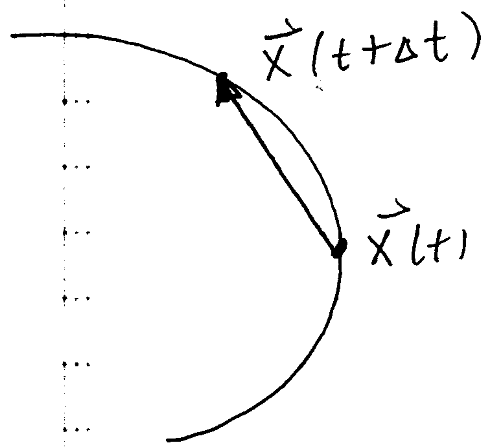
$\vec{X}(0) = \langle 1, 0, 0 \rangle$

The particle stays over the circle $x^2+y^2=1$ but rises in the z-direction.

Let $\vec{X}(t) = (x(t), y(t), z(t))$.

$\vec{X}(t+\Delta t) - \vec{X}(t)$

$= \big(x(t+\Delta t) - x(t),\ y(t+\Delta t) - y(t),\ z(t+\Delta t) - z(t)\big)$



From 1st semester calculus we have (assuming x, y, z are differentiable).

$$x(t+\Delta t) - x(t) = x'(t)\Delta t + o(\Delta t).$$

where

$o(\Delta t)$ <u>means</u>

<u>any</u> <u>quantity</u> depending on Δt s.t.

$$\frac{o(\Delta t)}{\Delta t} \to 0 \quad \text{as } \Delta t \to 0.$$

similarly

$$y(t+\Delta t) - y(t) = y'(t)\Delta t + o(\Delta t)$$

$$\vdots$$

$$z(t+\Delta t) - z(t) = z'(t)\Delta t + o(\Delta t)$$

So $$\vec{X}(t+\Delta t) - \vec{X}(t) = \Delta t\big(x'(t), y'(t), z'(t)\big) + \big(o(\Delta t), o(\Delta t), o(\Delta t)\big).$$

and

$$\therefore \quad \frac{\vec{X}(t+\Delta t) - \vec{X}(t)}{\Delta t}$$

$$= (x'(t), y'(t), z'(t))$$

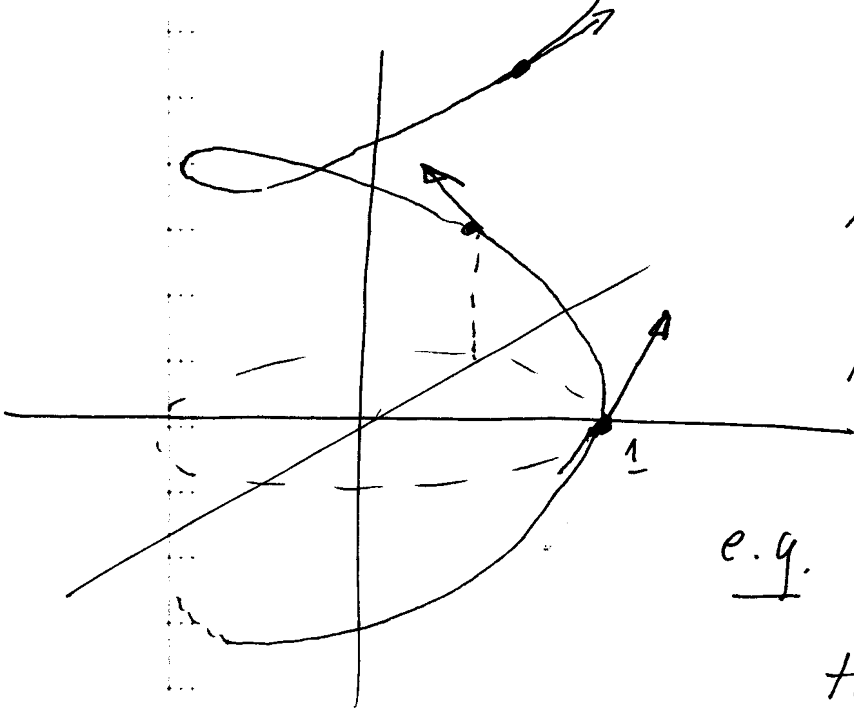
$$+ \quad \frac{1}{\Delta t}(o(\Delta t), o(\Delta t), o(\Delta t)).$$

We now see that

$$\lim \frac{\vec{X}(t+\Delta t) - \vec{X}(t)}{\Delta t} = (x'(t), y'(t), z'(t))$$

e.g. $\vec{X}(t) = (\cos t, \sin t, t)$

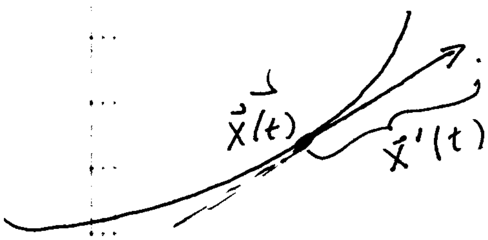
$\vec{X}'(t) = (-\sin t, \cos t, 1)$.



$\vec{X}'(0) = (0, 1, 1)$.

$\vec{X}'(\pi/2) = (-1, 0, 1)$

e.g. Find the line tangent to the curve $\vec{x}(t) = (\cos t, \sin t, t)$ at $t = \pi/4$.



$$\vec{X}(\pi/4) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \pi/4\right)$$

$$\vec{X}'(\pi/4) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right).$$

$$x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t$$
$$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t$$
$$z = \pi/4 + t$$

e.g. find the equation of the plane perpendicular to the curve

$\vec{X}(t) = (\cos t, \sin t, t)$

at $t = \pi/3$.

$\vec{X}(\pi/3) = \left(1/2, \frac{\sqrt{3}}{2}, \pi/3\right).$

$\vec{X}'(\pi/3) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right).$

$$-\frac{\sqrt{3}}{2}(x - 1/2) + \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + (z - \pi/3) = 0.$$

Speed

The speed of the curve $\vec{X}(t)$ is

$v(t) = \|\vec{X}'(t)\|.$

So $\quad v(t)^2 = \vec{X}'(t) \cdot \vec{X}'(t).$

e.g $\vec{X}(t) = (4\sin 2\pi t, 4\cos 2\pi t)$.

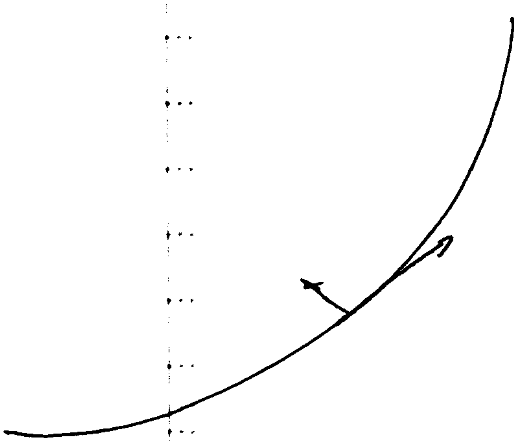
$\vec{X}'(t) = (8\pi\cos 2\pi t, -8\pi \sin 2\pi t)$.

$v(t) = 8\pi = \sqrt{(8\pi\cos 2\pi t)^2 + (-8\pi\sin 2\pi t)^2}$

Acceleration

Acceleration is a <u>vector</u>

$\vec{a}(t) = \frac{d}{dt}\left(\vec{X}'(t)\right) = \vec{X}''(t)$.



It has a component in the direction of the tangent vector and a component perpendicular to the tangent vector.

Projection of $\vec{a}(t)$ along $X'(t)$

is
$$\frac{\vec{X}''(t)\cdot X'(t)}{X'(t)\cdot X'(t)}\, X'(t).$$

The length of this projection is

$$\frac{|X''(t)\cdot X'(t)|}{\|X'(t)\|}.$$

Now
$$\frac{d}{dt}\big(X'(t)\cdot X'(t)\big) =$$