Math 32 9/14/09.

Last time

$\vec{X}(t) = (x(t), y(t), z(t))$

(with x, y, z smooth functions of t)

parameterizes
~~gives~~ a smooth curve in $\mathbb{R}^3$,

giving the position of a particle on the curve at time t.

$$\vec{X}'(t) = (x'(t), y'(t), z'(t))$$

$$= \lim_{\Delta t \to 0} \frac{\vec{X}(t+\Delta t) - \vec{X}(t)}{\Delta t}$$

gives the instantaneous velocity vector of the particle at time t.

This is also a tangent vector to the curve.

e.g. Find a parametric equation for the tangent line to the graph of $\vec{X}(t) = (\cos t, \sin t, e^t)$ at $t = \pi/3$.

$$\vec{X}'(t) = (-\sin t, \cos t, e^t)$$

$$\vec{X}'(\pi/3) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, e^{\pi/3}\right)$$

$$\vec{X}(\pi/3) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, e^{\pi/3}\right)$$

$$x = 1/2 - \frac{\sqrt{3}}{2} t$$

$$y = \sqrt{3}/2 + \frac{1}{2} t$$

$$z = e^{\pi/3} + e^{\pi/3} t.$$

What is the equation of the plane perpendicular to the curve at the same point?

$$-\frac{\sqrt{3}}{2}\left(x - 1/2\right) + \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + e^{\pi/3}\left(z - e^{\pi/3}\right) = 0.$$

Speed

$v(t) \equiv \| X'(t) \|$. (a scalar).

Acceleration

$\frac{d}{dt}\left[X'(t)\right] \equiv X''(t)$. (a vector).

Note that in general

$$\frac{d}{dt}\left(\|X'(t)\|\right) \neq \|X''(t)\|.$$

If $\vec{X}(t)$, $\vec{Y}(t)$ are two curves.

$\vec{X}(t) \cdot \vec{Y}(t)$ is a scalar function of t.

and $\frac{d}{dt}\left[\vec{X}(t)\cdot\vec{Y}(t)\right] = \vec{X}'(t)\cdot\vec{Y}(t) + \vec{X}(t)\cdot\vec{Y}'(t)$

pf: $\vec{X} = (x_1, x_2, x_3)$ $\vec{Y} = (y_1, y_2, y_3)$

$\vec{X}\cdot\vec{Y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

$\frac{d}{dt}(\vec{X}\cdot\vec{Y}) = \dot{x}_1 y_1 + x_1 \dot{y}_1 + \dot{x}_2 y_2 + x_2 \dot{y}_2 + \dot{x}_3 y_3 + x_3 \dot{y}_3$

$$= (\dot{x}_1, \dot{x}_2, \dot{x}_3) \cdot (y_1, y_2, y_3) + (x_1, x_2, x_3) \cdot (\dot{y}_1, \dot{y}_2, \dot{y}_3)$$

$$= \vec{X}'(t) \cdot \vec{Y}(t) + \vec{X}(t) \cdot \vec{Y}'(t)$$

$\blacksquare$

An important special case:

$$\cancel{\frac{d}{dt}\left(\|\vec{X}(t)\|^2\right) = \left(\frac{d}{dt}\right.}$$

$$\frac{d}{dt}\left(\|\vec{X}(t)\|^2\right) = \frac{d}{dt}\left(\vec{X}(t) \cdot \vec{X}(t)\right)$$

$$= 2\,\vec{X}'(t) \cdot \vec{X}(t)$$

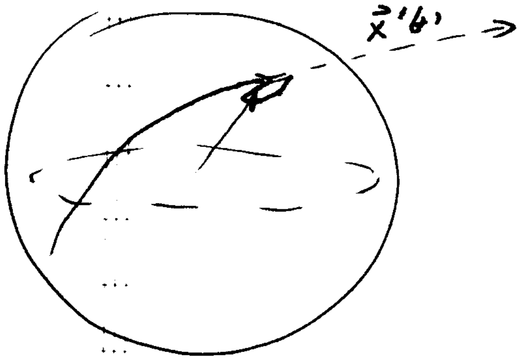
<u>e.g.</u> if $\|\vec{X}(t)\| = k$ then $\vec{X}(t)$ moves on a sphere of radius k centered at the origin.

We have $\|\vec{X}(t)\|^2 = k^2$ so

$$\frac{d}{dt}\|\vec{X}(t)\|^2 = 0$$

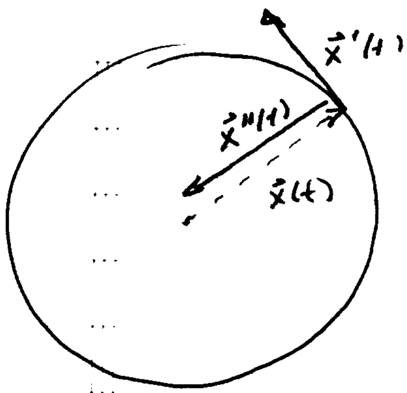
This is the same as $2\,\vec{X}'(t) \cdot \vec{X}(t) = 0$.

So the position vector and velocity vector are perpendicular



e.g. $\vec{X}(t) = (\cos t, \sin t)$

$\vec{X}'(t) = (-\sin t, \cos t)$



$\vec{X}''(t) = (-\cos t, -\sin t) = -\vec{X}(t).$

e.g. (#14 pg 60) Prove that if the acceleration of a curve is always perpendicular to its velocity, then its speed is constant.

Pf: We have

$$\vec{X}''(t) \cdot \vec{X}'(t) = 0 \quad \forall t.$$

But
$$2\vec{X}''(t) \cdot \vec{X}'(t) = \frac{d}{dt}\left(\vec{X}'(t) \cdot \vec{X}'(t)\right)$$
$$= \frac{d}{dt}\left(\|\vec{X}'(t)\|^2\right).$$

So $\|\vec{X}'(t)\|^2$ is constant

$\therefore$ $\|\vec{X}'(t)\|$ is constant.

e.g. Find the projection of the acceleration in the direction of the velocity of a curve $\vec{X}(t)$.

$$\frac{\vec{X}''(t) \cdot \vec{X}'(t)}{\vec{X}'(t) \cdot \vec{X}'(t)} \vec{X}'(t) = \frac{\vec{X}''(t) \cdot \vec{X}'(t)}{\|\vec{X}'(t)\|^2} \vec{X}'(t).$$

The length of this projection is

$$\frac{\vec{X}''(t) \cdot \vec{X}'(t)}{\|\vec{X}'(t)\|}.$$

Notice that

$$\frac{d}{dt}\left(\|\vec{x}\,'(t)\|^2\right) = 2\,\|\vec{x}\,'(t)\| \cdot \frac{d}{dt}\|x'(t)\|$$

so

$$\frac{d}{dt}\|x'(t)\| = \frac{x''(t)\cdot x'(t)}{\|\vec{x}\,'(t)\|}.$$

So the acceleration has a part (component) in the direction of the velocity whose length is the rate of change of the speed (along the curve). There is also (in general) a non-zero part of the acceleration which is perpendicular to the velocity.

Length of curves.

Interpreting the speed as the rate of change of the distance travelled it seems reasonable to define.

(For a curve $\vec{X}(t)$, $a \le t \le b$).

$$\int_a^b \|X'(t)\| \, dt.$$

to be the length of the curve.

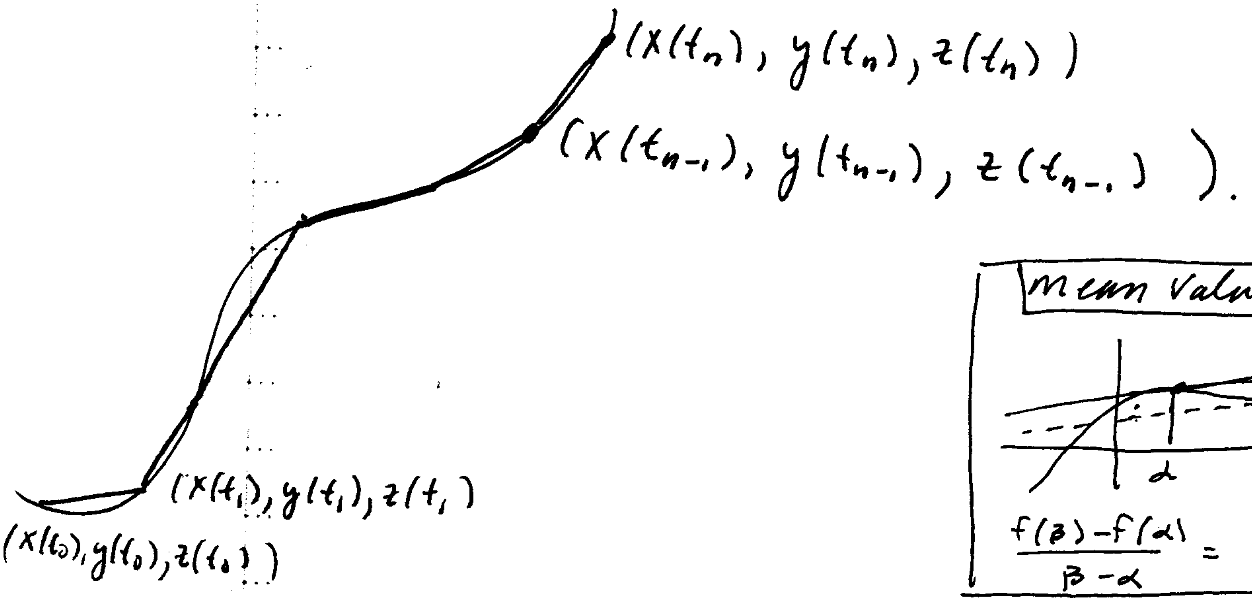
With $\vec{X}(t) = (x(t), y(t), z(t))$

this is

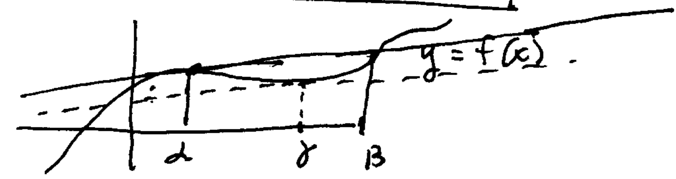
$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.$$

(Require that $X'(t) \neq 0 \quad \forall t \in [a,b]$ to rule out the particle doubling back over part of the curve).

Consider: $a = t_0$, $b = t_n$.



Mean Value Theorem



$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(\gamma) \text{ for some } \gamma,\ \alpha \le \gamma \le \beta.$$

$$\sum_{i=1}^{n} \sqrt{(x(t_i)-x(t_{i-1}))^2 + (y(t_i)-y(t_{i-1}))^2 + (z(t_i)-z(t_{i-1}))^2}$$

$$= \sum_{i=1}^{n} \sqrt{(x'(t_i^{*})(t_i-t_{i-1}))^2 + y'(t_i^{**})^2(t_i-t_{i-1})^2 + z'(t_i^{***})^2(t_i-t_{i-1})^2}$$

$$= \sum_{i=1}^{n} \sqrt{x'(t_i^{*})^2 + y'(t_i^{**})^2 + z'(t_i^{***})^2} \cdot (t_i - t_{i-1})$$

This is the length of a polygonal approximation to our curve and also a Riemann Sum for

$$\int_a^b \|x'(t)\|\, dt.$$