Math 32 9/16/09

From last time

if $\vec{X}'(t) \neq 0$ $\quad a \le t \le b$.

then the length of the curve $\vec{X}(t)$, $a \le t \le b$

is

$$\int_a^b \|X'(t)\| \, dt.$$

e.g. Find the length of the spiral

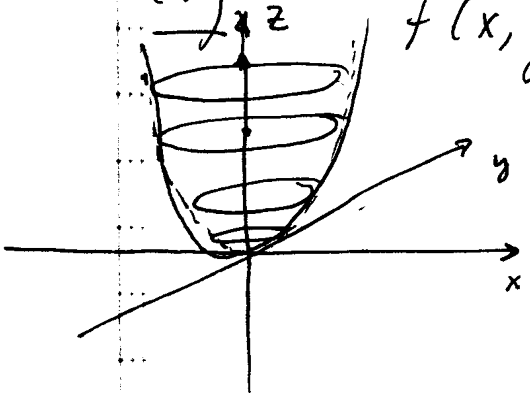
$$(\cos 4t, \sin 4t, t) \qquad 0 \le t \le \pi/8.$$

$$\int_0^{\pi/8} \sqrt{4\sin^2 4t + 4\cos^2 4t + 1} \, dt$$

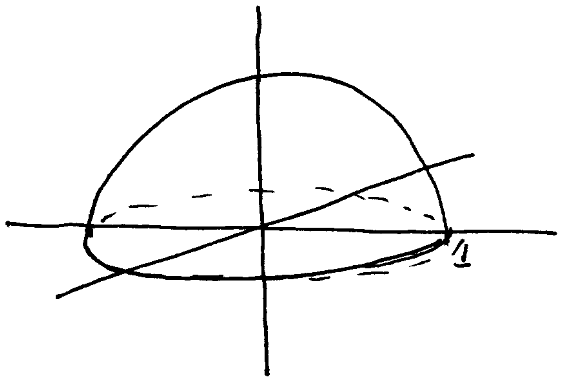
$$= \int_0^{\pi/8} 2\sqrt{2} \, dt = 2\sqrt{2} \cdot \frac{\pi}{8} = \frac{\sqrt{2}\,\pi}{4}.$$

Functions in $\mathbb{R}^2$ and $\mathbb{R}^3$.

e.g. $f(x,y) = x^2 + y^2 \; (= z)$



e.g. $f(x,y) = \sqrt{1-x^2-y^2}$



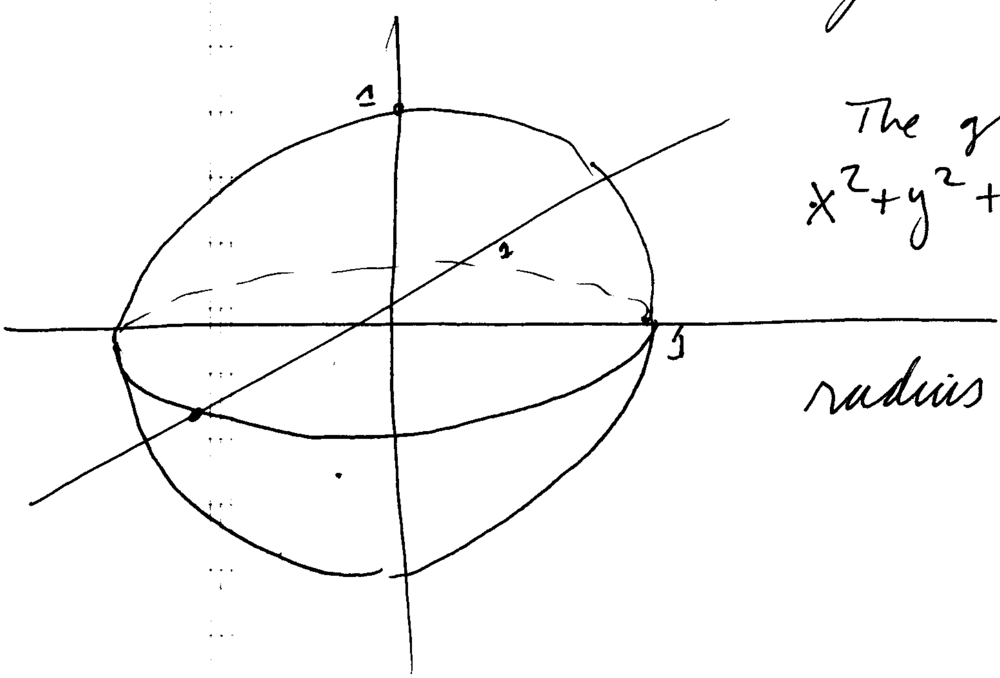
A half sphere.

e.g. $f(x,y,z) = x^2+y^2+z^2$

It's difficult to imagine a graph, but we can picture the "level ~~curves~~ surfaces"

$f(x,y,z) = c$ for various values of c.

(in this case, only $c \geq 0$ makes sense).

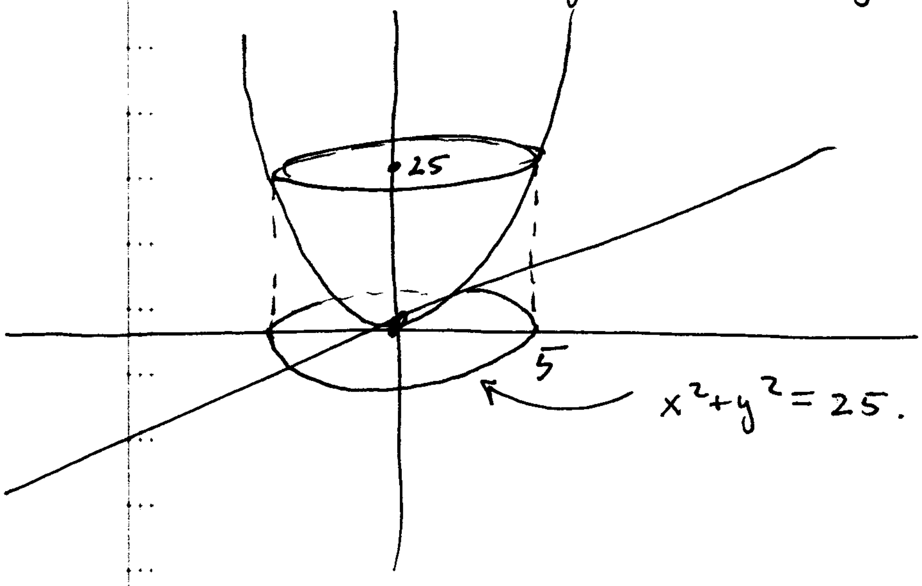


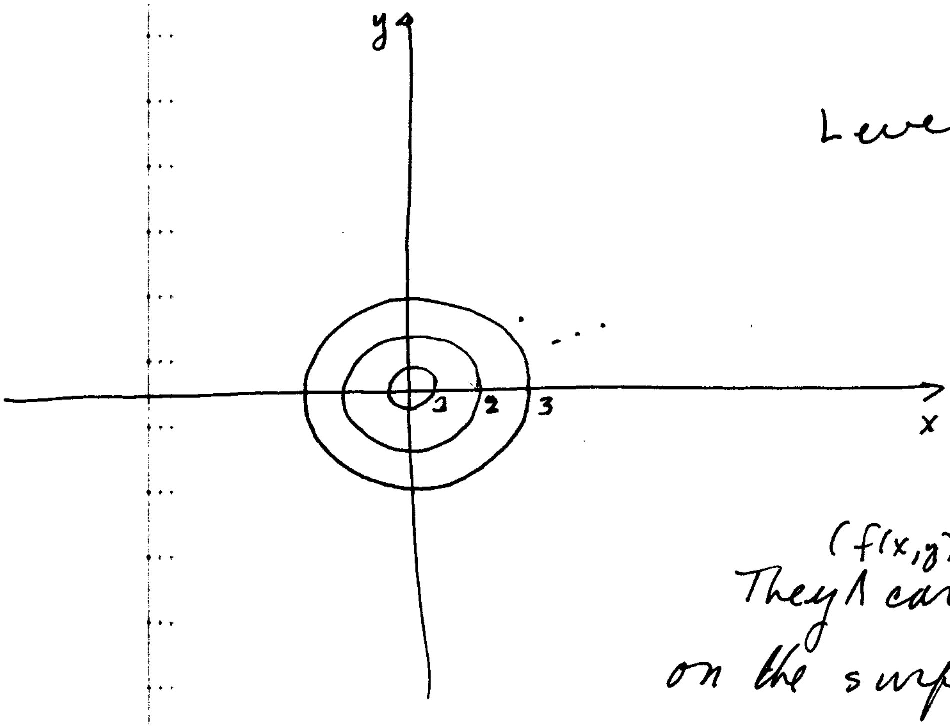
The graph of $x^2+y^2+z^2 = \underline{1}$ is the sphere of radius 1.

The notion of "level curve" is still useful in 2 dimensions.

e.g.

$$f(x,y) = x^2+y^2 = c.$$

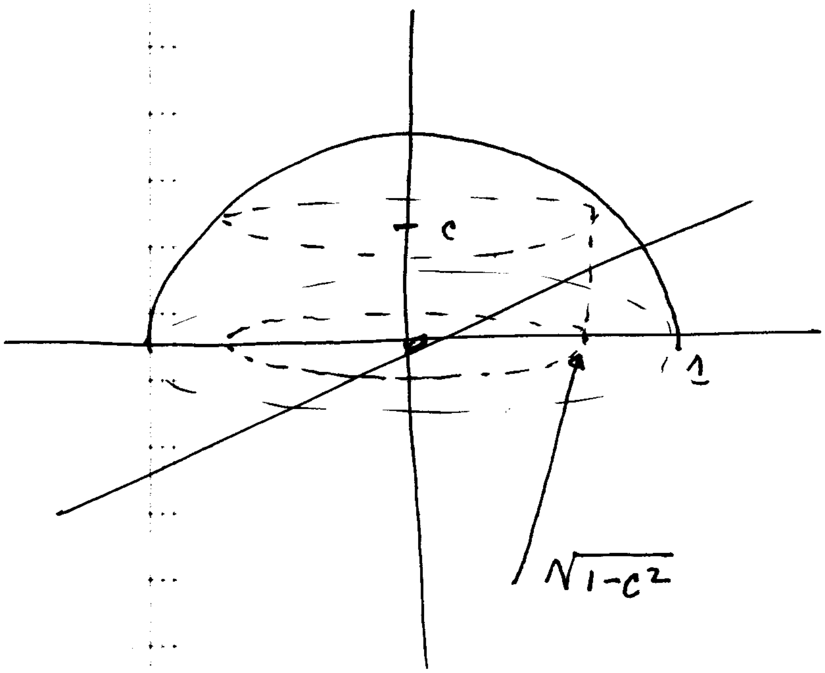




Level curves of $f(x,y) = x^2+y^2$ are circles in the plane.

They $(f(x,y)=c)$ correspond to curves on the surface at the height $z=c$.

e.g. $f(x,y) = \sqrt{1-x^2-y^2} = c \qquad 0 \le c \le 1.$



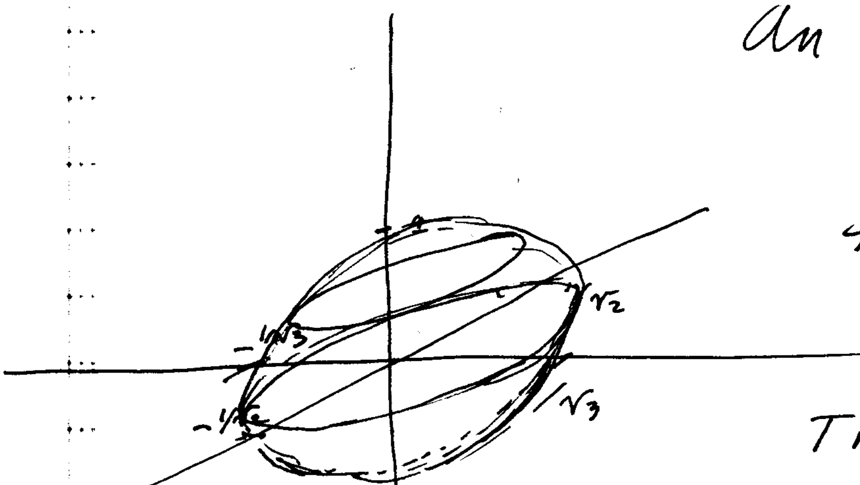
$$1 - x^2 - y^2 = c^2$$

$$x^2 + y^2 = \left(\sqrt{1-c^2}\right)^2$$

The curve $f(x,y) = c$ is a circle of radius $\sqrt{1-c^2}$.

e.g. $f(x,y,z) = 3x^2 + 2y^2 + z^2.$

An "ellipsoid".



$f(x,y,z) = 1.$

Try drawing the cross sections in the xy, yz and xz planes. to get a feel for the shape of the surface.

when $z=0$, (i.e. (in the x,y plane.)

the surface traces the curve

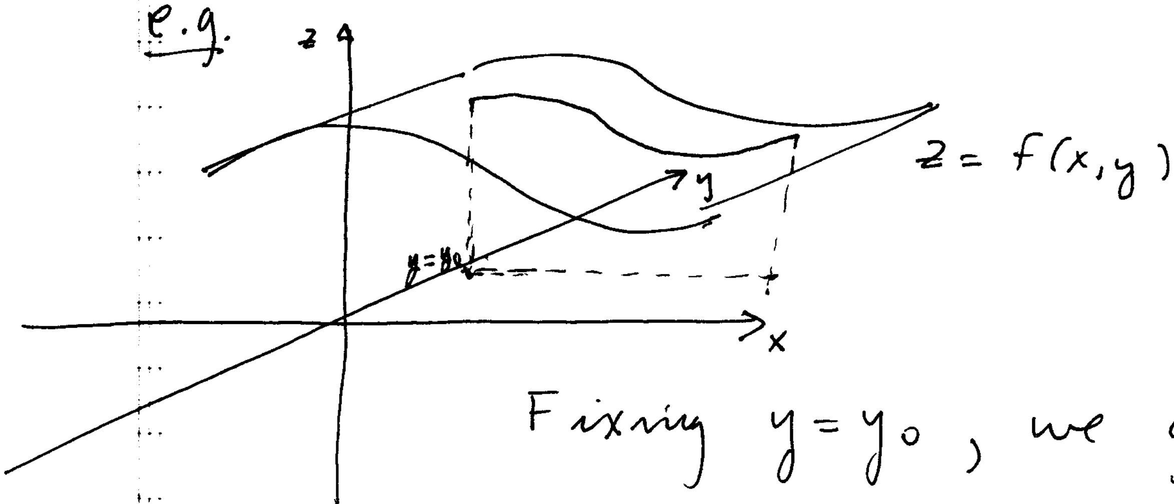
$$3x^2 + 2y^2 = 1.$$

$$\text{or} \quad \frac{x^2}{(1/\sqrt{3})^2} + \frac{y^2}{(1/\sqrt{2})^2} = 1.$$

$\vdots$

Partial derivatives

e.g.



Fixing $y = y_0$, we get a function of x only.

$$g(x) = f(x, y_0).$$

$g'(x)$ has the usual interpretation as a rate of change or slope of a tangent line.

and

$$g'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y_0) - f(x, y_0)}{\Delta x}.$$

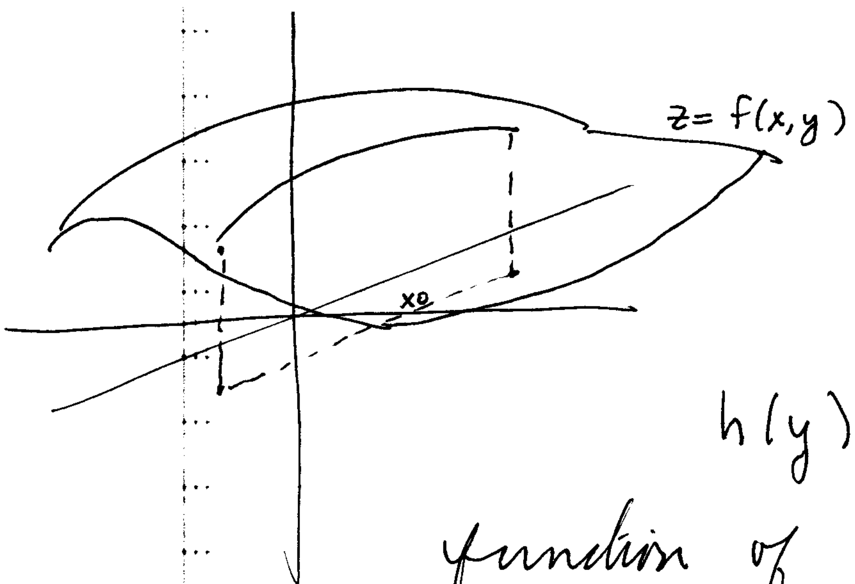
$$\equiv \frac{\partial f}{\partial x}(x, y_0).$$

"The partial derivative of f with respect to x".

Note that it is a function of both x and $y = y_0$.

Perhaps a more convenient notation is

$$D_1 f(x, y_0) \left(= \frac{\partial f}{\partial x}(x, y_0)\right).$$



Similarly, with $x = x_0$ fixed.

$h(y) = f(x_0, y)$ is a function of y only and we set $h'(y) = \frac{\partial f}{\partial x}(x_0, y) = \lim_{\Delta y \to 0} \frac{f(x_0, y+\Delta y) - f(x_0, y)}{\Delta y}$

All of the rules and tricks of differentiation we learned earlier carry through to the partial derivative setting. Just set all the other variables constant and differentiate

e.g. $f(x,y,z) = xyz\sin(x^2y^2+z^2)$.

$$\frac{\partial f}{\partial x} = yz\sin(x^2y^2+z^2) + xyz\cos(x^2y^2+z^2)\cdot 2xy^2$$

$$\frac{\partial f}{\partial y} = xz\sin(x^2y^2+z^2) + xyz\cos(x^2y^2+z^2)\cdot 2x^2y$$

$$\frac{\partial f}{\partial z} = xy\sin(x^2y^2+z^2) + xyz\cos(x^2y^2+z^2)\cdot 2z$$

Gradient

Two variables: $f(x,y)$

$$\vec{\nabla} f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right).$$

$\underbrace{\text{The gradient of } f}$ is a vector valued function of (x,y).

Three variables: $f(x,y,z)$

$$\vec{\nabla} f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$= \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right).$$

$\underline{\mathbb{R}^n}$: $\quad f(x_1, \ldots, x_n)$

$$\vec{\nabla} f(\underbrace{x_1, \ldots, x_n}_{\vec{x}}) = \left(D_1 f(\vec{x}), \ldots\ldots, D_n f(\vec{x})\right).$$

<u>e.g.</u> $$f(x,y,z) = e^{xyz}\cos(x+2y+z^2)$$

$$\frac{\partial f}{\partial x} = e^{xyz}\cdot yz\cos(x+2y+z^2)$$
$$- e^{xyz}\sin(x+2y+z^2).$$

$$\frac{\partial f}{\partial y} = e^{xyz}\cdot xz\cos(x+2y+z^2)$$
$$-2\cdot e^{xyz}\sin(x+2y+z^2)$$

$$\frac{\partial f}{\partial z} = e^{xyz}\cdot xy\cos(x+2y+z^2)$$
$$-2z\,e^{xyz}\sin(x+2y+z^2)$$

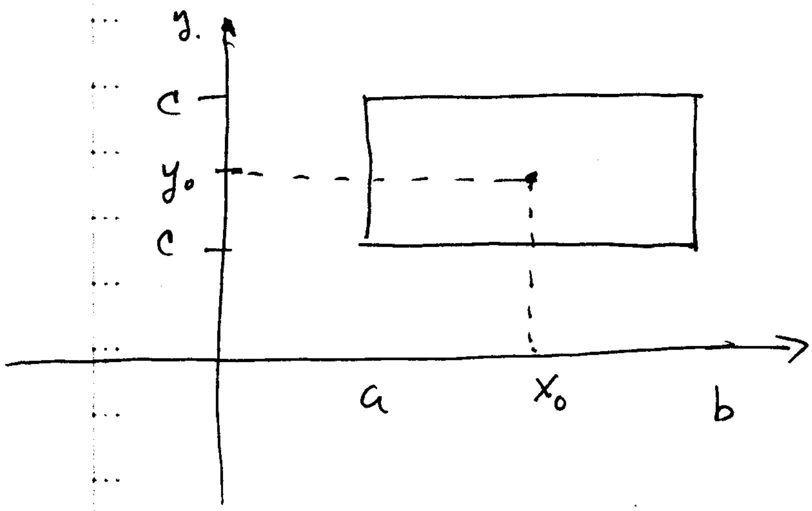
$$\vec{\nabla} f(\pi, 0, \sqrt{\pi})$$

$$= \left(0,\ \pi^{3/2},\ 0\right).$$

Differentiability and Gradient

Suppose $f(x,y)$ is defined for all (x,y) in a rectangle R

$$a \le x \le b \;, \quad c \le x \le d.$$



and has partial derivatives in R which are continuous.

Let (x_0, y_0) be the center of the rectangle (or any other point in R).

and consider

$$f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0, y_0).$$

for $\Delta x, \Delta y$ small enought that the 1st term is still defined.

We have (from 1 variable calculus).

$$g(x+\Delta x) = g(x) + (\Delta x) g'(x) + o(\Delta x).$$

(where $\frac{o(\Delta x)}{\Delta x} \to 0$ as $\Delta x \to 0$)

whenever g is differentiable at x.

So

$$f(x_0+\Delta x,\ y_0+\Delta y)$$

$$= f(x_0, y_0+\Delta y) + \frac{\partial f}{\partial x}(x_0, y_0+\Delta y)\,\Delta x$$

$$+ \underbrace{o(\Delta x)\left[(y+\Delta y_0)\right]}$$

$$\| $$

$$\Delta x \cdot \left[\frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)}{\Delta x} - \frac{\partial f}{\partial x}(x_0, y+\ldots\right.$$

$$\| $$

$\boxed{\text{$o(\Delta x)$ uniformly in Δy by continuity of the partials.}}$

$$= f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)\,\Delta y + o(\Delta y)$$

$$+ \left[\frac{\partial f}{\partial x}(x_0, y_0) + \epsilon(\Delta y)\right]\Delta x + o(\Delta x)\left[(y+\Delta y_0)\ldots\right.$$

$(\epsilon(\Delta y) \to 0$ as $\Delta y \to 0)$

$$= f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + o(\|(\Delta x, \Delta y)\|).$$

$$(*) \quad = f(x_0, y_0) + \vec{\nabla} f(x_0, y_0) \cdot (\Delta x, \Delta y) + o(\|(\Delta x, \Delta y)\|).$$

In analogy with the 1 dimensional situation

$$\left(g(x+\Delta x) = g(x) + g'(x)\Delta x + o(\Delta x) \right).$$

We say that $f(x,y)$ is differentiable at (x_0, y_0)

if $(*)$ holds at (x_0, y_0) for small enough $\|(\Delta x, \Delta y)\|$.

Notice that this gives an approxation of the function $f(x,y)$ near (x_0, y_0) which is the equation of a plane.

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + o\big(\|(x-x_0, y-y_0)\|\big).$$

$$z = z_0 + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

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A function $f(x)$, of one variable is differentiable at x iff

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{exists}$$

Then We write

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

This is the same as:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + \epsilon(\Delta x)$$

where $\epsilon(\Delta x) \to 0$ as $\Delta x \to 0$.

and

$\therefore$

$$f(x+\Delta x) - f(x) = f'(x)\Delta x + \epsilon(\Delta x)\Delta x$$

earlier, we called $\epsilon(\Delta x)\Delta x$, $o(\Delta x)$.

For a function of two variables, the analog would be to consider

$$f(\vec{x}+\Delta\vec{x}) = f(x+\Delta x, y+\Delta y).$$

but we don't form a difference quotient with $\Delta\vec{x}$ (a vector).

Instead we use the equation

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \epsilon(\Delta x)\Delta x$$

as the starting point.

Proceed one variable at a time:

$$f(x+\Delta x, y+\Delta y)$$

$$= f(x, y+\Delta y) + \frac{\partial f}{\partial x}(x, y+\Delta y)\,\Delta x + \epsilon(\Delta x, y+\Delta y)\,\Delta x$$

$$= f(x,y) + \frac{\partial f}{\partial y}(x,y)\Delta y + \epsilon(\Delta y)\Delta y$$

$$+ \left(\frac{\partial f}{\partial x}(x,y) + \epsilon(\Delta y)\right)\Delta x$$

$$+ \left(\epsilon(\Delta x) + \epsilon(\Delta y)\right)\Delta x.$$

$$= f(x,y) + \frac{\partial f}{\partial x}(x,y)\Delta x + \frac{\partial f}{\partial y}(x,y)\Delta y$$

$$+ \quad \epsilon(\Delta\vec{x})\,\|\Delta\vec{x}\|$$

We use the notation

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \text{"gradient of } f\text{"}$$

and get

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \vec{\nabla} f(x,y) \cdot (\Delta x, \Delta y)$$

$$+ \epsilon(\Delta\vec{x})\,\|\Delta\vec{x}\|$$

or $f(\vec{x} + \Delta\vec{x}) = f(\vec{x}) + \vec{\nabla} f(\vec{x}) \cdot \Delta\vec{x}$

$$+ \epsilon(\Delta\vec{x})\,\|\Delta\vec{x}\|.$$

in analogy to our one variable formula.

In one variable: (with $\Delta x = x - x_0$)

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \epsilon(x - x_0)\cdot(x - x_0)$$

and the 1st two terms give the equation of the tangent line to the graph of f at x_0.

In two variables: ($\Delta x = x - x_0$, $\Delta y = y - y_0$).

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$
$$+ \epsilon(\langle x - x_0, y - y_0 \rangle)\,\|\langle x - x_0, y - y_0 \rangle\|.$$

and the 1st three terms give the equation of the tangent plane to the graph of f at (x_0, y_0).

Higher order partial derivatives

e.g. $\quad f(x,y) = \cos(xy)$.

$$\frac{\partial f}{\partial x} = -y \sin(xy)$$

$$\frac{\partial f}{\partial y} = -x \sin(xy).$$

$$\frac{\partial^2 f}{\partial x^2} = D_1 D_1 f = -y^2 \cos(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = D_2 D_2 f = -x^2 \cos(xy)$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = D_2 D_1 f = -\sin(xy) - xy\cos(xy)$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = D_1 D_2 f = -\sin(xy) - xy\cos(xy)$$

Notice that

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y} \quad \text{in this case.}$$

This is also true more generally,

$$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \frac{\partial f}{\partial y}(x,y).$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) =$$

$$\lim_{\Delta x \to 0} \frac{\displaystyle\lim_{\Delta y \to 0} \frac{f(x+\Delta x, y+\Delta y - f(x+\Delta x, y)}{\Delta y} - \lim_{\Delta y \to 0}\frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \left[\frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x,y)}{\Delta x \, \Delta y}\right]$$

notice that the expression in square brackets does not change if we switch Δx and Δy.

so $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}(x,y)\right)$ is

$$\lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \left[\left(\begin{array}{c}\text{The same} \\ \text{thing}\end{array}\right). \right]$$

Many questions in advanced calculus boil down to asking whether the order of two limiting processes may be exchanged.

In this case, the answer is "yes", provided the functions f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ are all continuous (and exist...).

e.g. $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

find $D_1 D_2 D_3 f$:

$$D_3 f = -(x^2+y^2+z^2)^{-2} \cdot 2z$$

$$D_2 D_3 f = +4z\,(x^2+y^2+z^2)^{-3} \cdot 2y$$

$$D_1 D_2 D_3 f = -24zy\,(x^2+y^2+z^2)^{-4} \cdot 2x$$

$$= -48\,xyz\,(x^2+y^2+z^2)^{-4}$$

Does this equal $D_3 D_2 D_1 f$?

e.g. $f(x,y,z)$ satisfies

Laplace's equation if

$$\Delta f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

(we also say f is "harmonic").

Verify that

$f(x,y,z) = e^{3x+4y}\cos(5z)$ satisfies

Laplace's equation.

$$\frac{\partial f}{\partial x} = 3e^{3x+4y}\cos(5z)$$

$$\frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y}\cos(5z)$$

$$\frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y}\cos(5z)$$

$$\frac{\partial^{\cancel{2}} f}{\partial z^{\cancel{2}}} = -5e^{3x+4y}\sin(5z)$$

$$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y}\cos(5z)$$

$$\Delta f = (9+16-25)e^{3x+4y}\cos(5z).$$

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<u>Last time</u>

If $f: \mathbb{R}^3 \to \mathbb{R}$ has continuous partial derivatives (in a ball centered at (x,y,z)) then

$$f(x+\Delta x, y+\Delta y, z+\Delta z)$$

$$= f(x,y,z) + \frac{\partial f}{\partial x}(x,y,z)\Delta x + \frac{\partial f}{\partial y}(x,y,z)\Delta y + \frac{\partial f}{\partial z}(x,y,z)\Delta z$$

$$+ \|(\Delta x, \Delta y, \Delta z)\| \, \epsilon\big((\Delta x, \Delta y, \Delta z)\big).$$

Now suppose that x, y, z are (differentiable) functions of $t \in \mathbb{R}$.

$$g(t) = f(x(t), y(t), z(t)).$$

What is $g'(t)$?

$$g(t+\Delta t) = f(x(t+\Delta t), y(t+\Delta t), z(t+\Delta t))$$

$$= f\big(x(t) + x'(t)\Delta t + o(\Delta t),\ y(t) + y'(t)\Delta t + o(\Delta t),\ z(t) + z'(t)\Delta t + o(\Delta t)\big)$$

$$= f(x(t),y(t),z(t)) + \frac{\partial f}{\partial x}(x(t),y(t),z(t))\,x'(t)\Delta t + \frac{\partial f}{\partial y}(x(t),y(t),z(t))\,y'(t)\Delta t$$

$$+ \frac{\partial f}{\partial z}(x(t),y(t),z(t))\,z'(t)\Delta t \quad \boxed{+ \cdots \; o(\Delta t)}$$

$$\therefore\ g'(t) = \vec{\nabla} f(x(t), y(t), z(t)) \cdot \vec{X}'(t)$$

$$\left(\vec{X}(t) = (x(t), y(t), z(t)) \right)$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Suppose x, y, z are functions of u and v

$$g(u,v) = f(x(u,v), y(u,v), z(u,v))$$

(with ~~v~~ fixed, treat u as a single parameter like t)

e.g. $\cancel{\frac{\partial g}{\partial x}}$

$$\frac{\partial g}{\partial u} \cancel{\neq} \cancel{\frac{\partial f}{\partial x}}$$

$$\frac{\partial g}{\partial u}(u_0, v_0) = \frac{\partial f}{\partial x}(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)) \cdot \frac{\partial x}{\partial u}(u_0,v_0)$$

$$+ \frac{\partial f}{\partial y}(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)) \cdot \frac{\partial y}{\partial u}(u_0,v_0)$$

$$+ \frac{\partial f}{\partial z}(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)) \cdot \frac{\partial z}{\partial u}(u_0,v_0)$$

<u>pg91 #5.</u> f differentiable, $f: \mathbb{R}^2 \to \mathbb{R}$.

For some $m \geq 1$, an integer

$$f(tx, ty) = t^m f(x,y) \qquad \forall t, x, y.$$

Prove that

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = m f(x,y).$$

<u>Pf:</u>

$$\frac{d}{dt}\Big[f(tx, ty) \Big]_{t=t_0} = \cancel{x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}}$$

$$= x \frac{\partial f}{\partial x}(t_0 x, t_0 y) + y \frac{\partial f}{\partial x}(t_0 x, t_0 y)$$

$$\frac{d}{dt}\Big[t^m f(x,y) \Big]_{t=t_0} = m t_0^{m-1} f(x,y)$$

Put $t_0 = 1$. to get.

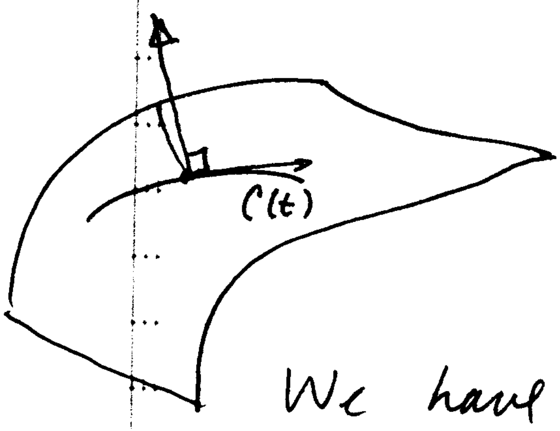
$$x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial x}(x,y) = m f(x,y).$$

Tangent Planes

Consider a level surface

$$f(x, y, z) = k$$ of some smooth function $f : \mathbb{R}^3 \to \mathbb{R}$.

At a point (x_0, y_0, z_0) in the surface where $\vec{\nabla} f(x_0, y_0, z_0) \neq 0$, take any smooth curve $C(t) = (x(t), y(t), z(t))$ lying in the surface which passes through (x_0, y_0, z_0) (say at time $t = t_0$).



We have

$$f(x(t), y(t), z(t)) = k$$

and $\vec{\nabla} f \cdot \vec{C}'(t) = 0$.

$\therefore$ $\vec{\nabla} f(x_0, y_0, z_0)$ is perpendicular to $\vec{C}'(t_0)$.

Since $C(t)$ was an arbitrary smooth curve contained in the surface, passing through (x_0, y_0, z_0)

We define the tangent plane to the surface at (x_0, y_0, z_0) to be the plane at (x_0, y_0, z_0) with normal vector $\vec{\nabla} f(x_0, y_0, z_0)$.

e.g. Find the equation of the tangent plane to the surface

$$x^2 + xy^2 + y^3 + z + 1 = 0 \text{ at } (2, -3, 4)$$

$$f(x, y, z) = x^2 + xy^2 + y^3 + z + 1$$

$$\frac{\partial f}{\partial x} = 2x + y^2 \qquad \left.\frac{\partial f}{\partial x}\right|_{(2,-3,4)} = 13$$

$$\frac{\partial f}{\partial y} = 2xy + 3y^2 \qquad \left.\frac{\partial f}{\partial y}\right|_{(2,-3,4)} = 15$$

$$\frac{\partial f}{\partial z} = 1. \qquad \left.\frac{\partial f}{\partial z}\right|_{(2,-3,4)} = 1.$$

$$\boxed{13(x-2) + 15(y+3) + (z-4) = 0}$$

Equation of the normal line?

$$x = 2 + 13t$$
$$y = -3 + 15t$$
$$z = 4 + t.$$

Pg. 98 #13.

a) $C(t)$ a diff. curve

lies in

$f(x,y,z) \equiv x^2 + 4y^2 + 9z^2 = 14$

s.t. $C(0) = (1,1,1)$

$h(t) \equiv f(C(t))$

Find $h'(0)$.

$h'(t) = \vec{\nabla} f(\vec{C}(t)) \cdot \vec{C}'(t) = 0.$

since $h(t)$ is constant.

b.) $g(x,y,z) = x^2 + y^2 + z^2$

$k(t) = g(C(t)) \qquad C'(0) = (4,-1,0)$

$k'(t) = \vec{\nabla g} \cdot \vec{C}'(t)$

$k'(0) = \vec{\nabla g}(x(0), y(0), z(0)) \cdot \vec{C}'(0)$

$= \vec{\nabla g}(1,1,1) \cdot (4,-1,0)$

$= \langle 2,2,2 \rangle \cdot (4,-1,0) = 6.$