Math 32 9/30/09.

<u>Def'n</u> $U \subset \mathbb{R}^3$ is <u>open</u> if for each $\vec{x} \in U$ there is a ball of some radius $r>0$ (r depends on x) centered at x which is also contained in U.

$$\exists\, r>0 \quad \text{s.t} \quad B_r(\vec{x}) \subset U$$

$$\text{where} \quad B_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^3 : \|\vec{x}-\vec{y}\| < r\}.$$

The definition is the same in $\mathbb{R}^2$.

<u>Def'n</u> An open set $U \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is <u>connected</u> if for any two points $\vec{P}, \vec{Q}$ in U, there is a differentiable curve in U joining $\vec{P}$ to $\vec{Q}$

i.e. there is $\vec{C}(t)$, a smooth curve, with $C(\vec{t_1}) = \vec{P}$ $\vec{C}(t_2) = \vec{Q}$ and $\vec{C}(t) \in U$ for all t with $t_1 \le t \le t_2$

<u>Thm:</u> Let U be a connected open set. Let f, g be two differentiable functions on U.

if $\vec{\nabla f} = \vec{\nabla g}$ on U then there is a constant k so that

$$f(\vec{x}) = g(\vec{x}) + k \quad \text{for all } \vec{x} \in U.$$

Pf: if $\nabla \vec{f} = \nabla \vec{g}$ on U then with $h = f - g$ we have $\nabla \vec{h} = 0$ and we will show that h is constant. Let $\vec{P}, \vec{Q}$ be any two points of U. and let $\vec{C}(t)$ be a curve in U which joins them. Then

$$h(\vec{Q}) - h(\vec{P}) = h(C(t_2)) - h(C(t_1))$$

$$= \int_{t_1}^{t_2} \frac{d}{dt}\left(h(C(t))\right) dt.$$

$$= \int_{t_1}^{t_2} \nabla h(C(t)) \cdot \vec{C'}(t)\, dt.$$

$$= 0.$$

$\therefore$ h is constant. ▨

This shows that when a vector field (in a connected open set) has a potential function, the only other potential functions differ by a constants

When does a vector field $\vec{F}$ have a potential function?

<u>A necessary condition in $\mathbb{R}^2$:</u> Suppose f, g are differentiable in U with continuous derivatives.

if $\quad F = \langle f, g \rangle = \nabla \varphi = \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \rangle$ in U

then $\quad f = \frac{\partial \varphi}{\partial x} \quad$ and $\quad g = \frac{\partial \varphi}{\partial y} \quad$ in U.

and $\quad \frac{\partial}{\partial y} f = \frac{\partial^2}{\partial y \partial x} \varphi \qquad \frac{\partial g}{\partial x} = \frac{\partial^2}{\partial x \partial y} \varphi.$

Since f and g are differentiable with cont. derivatives we have $\quad D_2 D_1 \varphi = D_1 D_2 \varphi$

so $\quad \boxed{\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.}$

<u>Thm:</u> if f, g are differentiable in an open set U with continuous derivatives but $\quad \frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x} \quad$ in U

then $\quad \vec{F} = \langle f, g \rangle \quad$ is not conservative.

e.g. $\left(\frac{1}{x},\ x e^{xy}\right)$ is a smooth vector field in $\mathbb{R}^2 \setminus \{\vec{0}\} \equiv U$

$$\frac{\partial}{\partial y}\left(\frac{1}{x}\right) = 0 \qquad \frac{\partial}{\partial x}\left(x e^{x}y\right) = \left(e^{x} + x e^{x}\right) y.$$

$$\neq 0 \text{ in } U$$

So $\vec{F} = \left(\frac{1}{x}, x e^{xy}\right)$ is not conservative.

e.g. $\left(\frac{x}{\sqrt{x^2+y^2}},\ 3xy^2\right)$ is smooth in

$$\mathbb{R}^2 \setminus \{0\} \equiv U.$$

$$\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{1}{2} x \cdot (x^2+y^2)^{-3/2} \cdot 2y$$

$$= \frac{-xy}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial}{\partial x}\left(3xy^2\right) = 3y^2.$$

So $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, 3xy^2\right)$ is not conservative.

Our <u>necessary</u> <u>condition</u> $\left\{ \begin{array}{l} f \text{ and } g \text{ are} \\ \text{assumed to have} \\ \text{continuous derivatives} \end{array} \right.$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

for $\langle f, g \rangle$ to be the gradient of some potential function becomes a <u>sufficient</u> <u>condition</u> provided that U is an open set with "no holes". For example, if U is a rectangle or the whole plane this ~~on~~ "no holes" condition is satisfied.

<u>Thm</u>: f, g are diff. with cont. derivatives on U. U is either an open rectangle or the whole plane.

If $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on U then

$\vec{F} = \langle f, g \rangle$ is conservative.

We will prove the theorem in a few steps later. For now we will see how it can be used.

e.g. $\vec{F}(x,y) = (2xy,\ x^2+3y^2)$

$$\frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial}{\partial x}(x^2+3y^2) = 2x$$

So the theorem tells us that $\vec{F} = \nabla\vec{\varphi}$ for some function φ.

We must have.

$$\frac{\partial \varphi}{\partial x} = 2xy.$$

$$\frac{\partial \varphi}{\partial y} = x^2+3y^2.$$

Integrate the 1st equation:

$$\varphi(x,y) = x^2 y + h(y)$$

Notice that the "constant of integration" is a function of y.

We now require this φ to satisfy the 2nd equation

$$\frac{\partial}{\partial y}\left(x^2 y + h(y)\right) = x^2 + h'(y) = x^2 + 3y^2.$$

For equality, we need

$$h'(y) = 3y^2$$

or $h(y) = y^3 + C.$

We see now that

$$\varphi(x,y) = x^2 y + y^3 + C$$

is a potential function (for any constant C).

e.g.

Pg 193

#10 a) $\vec{F}\ (ye^{xy},\ xe^{xy})$

$$\frac{\partial}{\partial y}(ye^{xy}) = e^{xy} + xye^{xy}$$

$$\frac{\partial}{\partial x}(xe^{xy}) = e^{xy} + xye^{xy}.$$

The vector field is conservative.

To find φ s.t. $\nabla\varphi = \vec{F}$:

$$\frac{\partial\varphi}{\partial x} = ye^{xy}$$

$$\frac{\partial\varphi}{\partial y} = xe^{xy}$$

$$\varphi(x,y) = e^{xy} + h(y)$$

$$\frac{\partial}{\partial y}(e^{xy} + h(y)) = xe^{xy} + h'(y) = xe^{xy}.$$

$$h'(y) = 0 \quad \text{so} \quad h(y) = C$$

$$\varphi(x,y) = e^{xy} + C.$$

10b). $\vec{F} = (y \cos xy\ ,\ x \cos xy)$

$$\frac{\partial}{\partial y}(y \cos xy) = \cos xy + xy \cos xy.$$

$$\frac{\partial}{\partial x}(x \cos xy) = \cos xy + xy \cos xy.$$

$\vec{F}$ is conservative.

For $\nabla \vec{\varphi} = \vec{F}$ we need.

$$\frac{\partial \varphi}{\partial x} = y \cos xy.$$

$$\frac{\partial \varphi}{\partial y} = x \cos xy.$$

$$\varphi(x,y) = \sin(xy) + h(y).$$

$$\frac{\partial}{\partial y}(\sin(xy) + h(y)) = x \cos xy + h'(y). = x \cos xy.$$

$h'(y) = 0$ so $h(y) = C.$

$$\varphi(x,y) = \sin(xy) + C.$$

10c) $\vec{F} = \langle 2xy \cos x^2y \, , \, x^2 \cos x^2 y \rangle$

$$\frac{\partial}{\partial y}\left(2xy \cos x^2 y\right) = 2x \cos x^2 y - 2x^3 y \sin(x^2 y)$$

$$\frac{\partial}{\partial x}\left(x^2 \cos x^2 y\right) = 2x \cos x^2 y - 2x^3 y \sin(x^2 y)$$

$\vec{F}$ is conservative

For $\nabla \varphi = \vec{F}$ we need.

$$\frac{\partial \varphi}{\partial x} = 2xy \cos x^2 y$$

$$\frac{\partial \varphi}{\partial y} = x^2 \cos x^2 y$$

$$\varphi(x,y) = \sin(x^2 y) + h(y).$$

$$\frac{\partial}{\partial y}\left(\sin(x^2 y) + h(y)\right) = x^2 \cos(x^2 y) + h'(y).$$

$$h'(y) = 0 \Rightarrow h = C.$$

$$\varphi(x,y) = \sin(x^2 y) + C$$

Math 32 10/2/09

Existence of potential functions

Thm: Let $\vec{F} = (f_1, f_2, f_3)$ be a smooth vector field defined on an open rectangular box in $\mathbb{R}^3$ such that

$$D_i f_j = D_j f_i \qquad 1 \le i, j \le 3$$

i.e. $D_1 f_2 = D_2 f_1$, $D_1 f_3 = D_3 f_1$, $D_2 f_3 = D_3 f_2$.

Then $\vec{F}$ is conservative

We again postpone the proof and consider the practical question of how to find a potential function when the theorem applies.

e.g. $\vec{F}(x,y,z) = (y\cos(xy),\ x\cos(xy) + 2yz^3,\ 3y^2z^2)$.

$D_1 f_2 = -xy\sin(xy) \qquad D_1 f_3 = 0 \qquad D_2 f_3 = 6yz^2$

$D_2 f_1 = -xy\sin(xy) \qquad D_3 f_1 = 0 \qquad D_3 f_2 = 6yz^2$

$$\frac{\partial \varphi}{\partial x} = y\cos(xy)$$

$$\frac{\partial \varphi}{\partial y} = x\cos(xy) + 2yz^3$$

$$\frac{\partial \varphi}{\partial z} = 3y^2z^2$$

Integrating the 1st equation w.r.t. x gives

$$\varphi(x,y,z) = \sin(xy) + h(y,z)$$

To satisfy the 2nd equation we need:

$$\frac{\partial}{\partial y}\left(\sin(xy) + h(y,z)\right) = x\cos(xy) + \frac{\partial h}{\partial y}(y,z)$$

$$= x\cos(xy) + 2yz^3$$

or $$\frac{\partial h}{\partial y}(y,z) = 2yz^3.$$

Integrating this, in y, gives

$$h(y,z) = y^2z^3 + k(z)$$

and $$\varphi(x,y,z) = \sin(xy) + y^2z^3 + k(z)$$

to satisfy the 3rd equation we need

$$\frac{\partial \varphi}{\partial z} = 3y^2z^2 + k'(z) = 3y^2z^2$$

so $k'(z) = 0$. and we have.

$$\varphi(x,y,z) = \sin(xy) + y^2z^3 + C$$

for the potential.

<u>e.g.</u> (Pg 193 #13 i))

$$\vec{F} = (y^3z + y,\ 3xy^2z + x + z,\ xy^3 + y).$$

$D_1 f_2 = 3y^2z + 1$ $\qquad$ $D_2 f_3 = 3xy^2 + 1$

$D_2 f_1 = 3y^2z + 1$ $\qquad$ $D_3 f_2 = 3xy^2 + 1$

$D_3 f_1 = y^3$

$D_1 f_3 = y^3$

$\vec{F}$ is conservative.

$$\frac{\partial \varphi}{\partial x} = y^3z + y. \implies \varphi(x,y,z) = x(y^3z + y) + h(y,z).$$

$$\frac{\partial}{\partial y}\left(xy^3z + xy + h(y,z)\right) = 3y^2xz + x + \frac{\partial h}{\partial y}(y,z)$$

$\Rightarrow$ (by equation for $\frac{\partial \varphi}{\partial y}$)

$$\frac{\partial h}{\partial y}(y,z) = z$$

so $h(y,z) = yz + k(z)$.

and $\varphi(x,y,z) = xy^3z + xy + yz + k(z)$.

the equation for $\frac{\partial \varphi}{\partial z}$.

$\Rightarrow$ $k'(z) = 0$ so

$$\varphi(x,y,z) = xy^3z + xy + yz + C.$$

is a potential function.

Divergence and Curl

e.g. pg 193 #15. $\vec{F} = (f_1, f_2, f_3)$

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}f_3 - \frac{\partial}{\partial z}f_2\right) - \vec{j}\left(\frac{\partial}{\partial x}f_3 - \frac{\partial}{\partial z}f_1\right) + \vec{k}\left(\frac{\partial}{\partial x}f_2 - \frac{\partial}{\partial y}f_1\right)$$

$$= \vec{\nabla} \times \vec{F}$$

$$\operatorname{div} \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (f_1, f_2, f_3)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \vec{\nabla} \cdot \vec{F}.$$

a) $\operatorname{div}(\operatorname{curl} \vec{F}) = 0.$ ($\vec{F}$ is smooth)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f_3 - \frac{\partial}{\partial z} f_2\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} f_1 - \frac{\partial}{\partial x} f_3\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} f_2 - \frac{\partial}{\partial y} f_1\right)$$

$$= 0.$$

b) $\operatorname{curl}(\vec{\nabla}\varphi) = 0.$ for any smooth φ.

$$\vec{i}\left(\frac{\partial}{\partial y}\frac{\partial}{\partial z}\varphi - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\varphi\right) + \vec{j}\left(\frac{\partial}{\partial z}\frac{\partial}{\partial x}\varphi - \frac{\partial}{\partial x}\frac{\partial}{\partial z}\varphi\right)$$

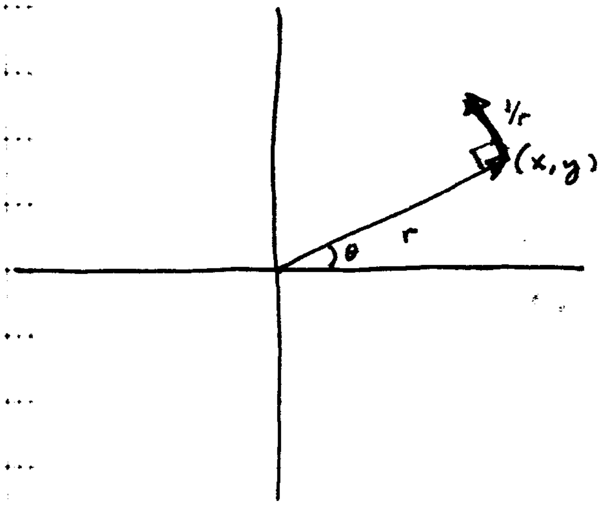
$$+ \vec{k}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\varphi - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\varphi\right)$$

$$= \vec{0}.$$

Math 32 10-7-09.

Vector Fields, Potential Functions cont.

e.g. $\vec{G}(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$

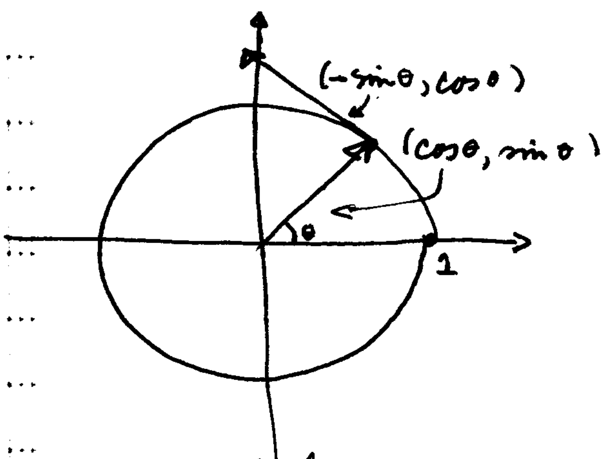


$x = r\cos\theta$

$y = r\sin\theta$

$$G(x,y) = \left(-\frac{r\sin\theta}{r^2}, \frac{r\cos\theta}{r^2} \right)$$

$$= \frac{1}{r}(-\sin\theta, \cos\theta)$$



Notice that

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2-2x\cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

So, by our still unproved theorem $\vec{G}$ has a potential function in any rectangle not containing the origin

$$\frac{\partial \varphi}{\partial x} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \varphi}{\partial y} = \frac{x}{x^2+y^2}$$

$$\varphi(x,y) = \int \frac{-y}{x^2+y^2}\,dx = -\int \frac{y}{x^2}\frac{1}{1+\left(\frac{y}{x}\right)^2}\,dx$$

$$\boxed{u = \frac{y}{x} \qquad du = -\frac{y}{x^2}\,dx}$$

$$= \int \frac{du}{1+u^2}$$

$$= \arctan\left(\frac{y}{x}\right) + h(y)$$

$$\frac{\partial}{\partial y}\left(\arctan\left(\frac{y}{x}\right) + h(y)\right)$$

$$= \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} + h'(y)$$

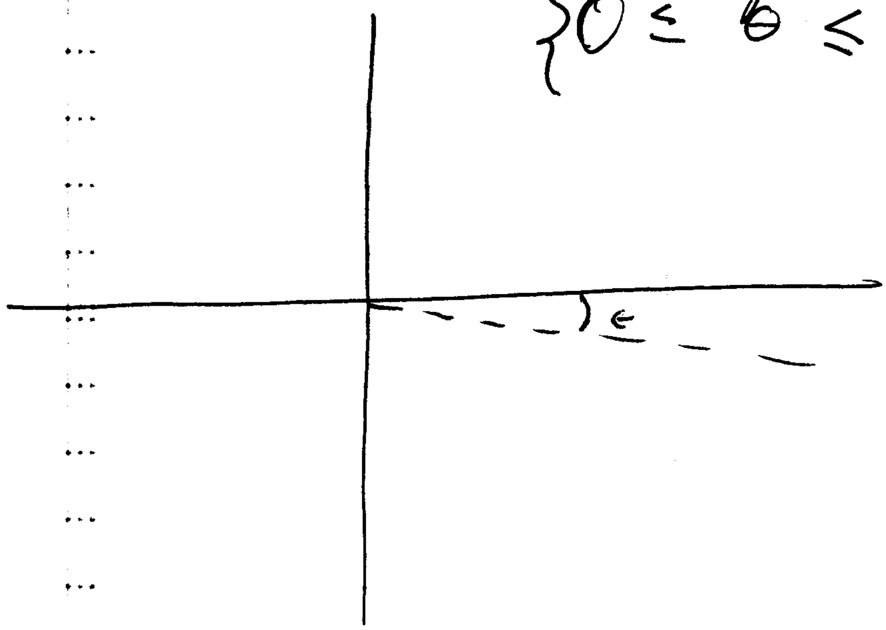
$$= \frac{x}{x^2+y^2} + h'(y) = \frac{x}{x^2+y^2}$$

so $h'(y) = 0$ and h is constant

$\therefore \quad \varphi(x,y) = \arctan\left(\frac{y}{x}\right)$

gives a potential function in any rectangle which does not meet the line $x=0$.

In fact, in any region of the form $\{0 \le \theta \le 2\pi - \epsilon,\ r>0\}$



we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

so ($\frac{\partial}{\partial y}$ of 1st, $\frac{\partial}{\partial x}$ of 2nd).

$$0 = \frac{\partial r}{\partial y}\cos\theta - r\sin\theta\,\frac{\partial\theta}{\partial y}.$$

$$= \frac{y}{r}\,\frac{x}{r} - y\,\frac{\partial\theta}{\partial y} \Rightarrow \frac{\partial\theta}{\partial y} = \frac{x}{r^2}$$

and

$$0 = \frac{\partial r}{\partial x}\sin\theta + r\cos\theta\,\frac{\partial \theta}{\partial x}$$

$$= \frac{x}{r}\,\frac{y}{r} + x\,\frac{\partial \theta}{\partial x}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-y}{r^2}.$$

So θ gives a potential function in any such region.

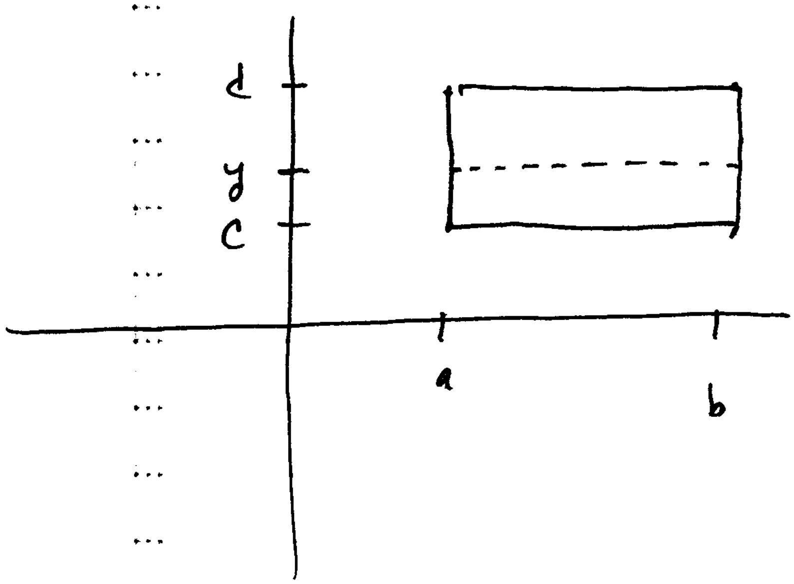
Later we will see that for this vector field $\vec{G}$, no potential function can be defined in $\mathbb{R}^2 \setminus \{\vec{0}\}$.

This is already plausible since the value of θ jumps by 2π when crossing $\{x \geq 0\}$.

To give a proof of our theorem on the existence of potential functions, we will need to differentiate under an integral sign:

Suppose f is continuous on

$$a \le x \le b$$
$$c \le y \le d.$$



Define
$$\psi(y) = \int_a^b f(x,y)\,dx$$

If $\frac{\partial f}{\partial y}$ exists and is continuous in the same rectangle the

we claim that

$$\psi'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

e.g. $f(x,y) = \sin(xy)$

$$\psi(y) = \int_0^{\pi} \sin(xy)\,dx$$

$$= -\frac{1}{y}\cos(xy)\Big|_{x=0}^{x=\pi}$$

$$= -\frac{1}{y}\cos(\pi y) + \frac{1}{y} = -\left(\frac{\cos(\pi y) - 1}{y}\right)$$

Now $$\psi'(y) = \frac{(\pi \sin \pi y)\,y - (1 - \cos \pi y)}{y^2}$$

$$= \frac{\pi \sin \pi y}{y} - \frac{1 - \cos \pi y}{y^2}$$

and

$$\int_0^{\pi} \frac{\partial}{\partial y}\left[\sin(xy)\right] dx$$

$$= \int_0^{\pi} x \cos(xy)\,dx$$

integrating by parts with $u = x$, $dv = \cos(xy)\,dx$

we see that $\begin{pmatrix} u = x & dv = \cos(xy)\,dx \\ du = dx & v = \frac{1}{y}\sin(xy) \end{pmatrix}$

$$\int_0^\pi x\cos(xy)\,dx = \frac{x}{y}\sin(xy)\Big|_0^\pi - \frac{1}{y}\int_0^\pi \sin(xy)\,dx$$

$$= \frac{\pi\sin(\pi y)}{y} - \frac{1}{y}\left(-\frac{\cos(xy)}{y}\Big|_0^\pi\right)$$

$$= \frac{\pi\sin(\pi y)}{y} - \frac{1}{y}\left(\frac{1}{y} - \frac{\cos\pi y}{y}\right)$$

$$= \frac{\pi\sin\pi y}{y} - \frac{1-\cos\pi y}{y^2}$$

To see why the trick works in general

$$\left(\psi'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx \right)$$

consider that

$$\frac{\psi(y+\Delta y) - \psi(y)}{\Delta y} = \int_a^b \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}\,dx.$$

This shows that

$$\psi'(y) = \lim_{\Delta y \to 0} \int_a^b \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}\,dx.$$

So our claim holds if we can pass the $\lim_{\Delta y \to 0}$ under the integral.

This is really an exchange of the order of two limiting processes.

(What is the other one?).

This exchange is justified when both limits exist "uniformly"

and ~~thus~~ for this, it is enough to know that $D_2 f$ exists and is continuous on the rectangle.