RANDOM WALK AND BOUNDARY BEHAVIOR OF FUNCTIONS IN THE DISK.

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ABSTRACT. Simple martingale proofs of some results of Rohde [9, 10] on the boundary behavior of Bloch functions are presented, making clear their connection with random walk in the plane.

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1. Introduction

A function f, defined and analytic in the unit disk, is called a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

We write $f \in \mathcal{B}$. The function is said to be in the little Bloch space $\mathcal{B}_0$ if

$$(1-|z|^2)|f'(z)| \to 0 \qquad |z| \to 1 \quad z \in \mathbb{D}.$$

The following proposition, which establishes a close connection between Bloch functions and conformal mappings, is well known.(See [1, 2])

Proposition 1.1. *If g is a univalent function in $\mathbb{D}$ and $f = \log g'$ then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \le 6$. Conversely, if $\|f\|_{\mathcal{B}} \le 1$ then there exists a univalent function g such that $f = \log g'$.*

In this note we use the device of Bloch martingales, developed by Makarov in [6] and briefly outlined below, to give simple proofs of the following two theorems of S. Rohde.

Theorem 1.2 (Rohde)**.** *An inner function in the little Bloch space which is not a finite Blaschke product has, for each $\delta \in \mathbb{D}$, the non-tangential limit δ on a set $E_\delta \subset \mathbb{T}$ with Hausdorff dimension 1.*

Theorem 1.3 (Rohde)**.** *Let $\{Q_n\}$ be a sequence of squares in the plane with pairwise disjoint interiors, edges parallel to the coordinate axes and of length*

$a > 0$, and such that Q_n is adjacent to Q_{n+1} for all n. There is a universal constant $K > 0$ such that if $g \in \mathcal{B}$ has nontangential limits almost nowhere then there exists a set E with

$$\text{dim } E \geq 1 - a^{-1}K\|g\|_{\mathcal{B}}$$

such that, for each $\zeta \in E$ we can find $r_n \to 1$ with

$$g(r\zeta) \subset Q_n \cup Q_{n+1} \qquad \text{for } r_n \leq r \leq r_{n+1}.$$

Given an arc $I \subset \mathbb{D}$ let $z_I = \tau(0)$ where τ is the conformal self mapping of the disk which maps $\partial\mathbb{D} \cup \{\text{Re } z > 0\}$ onto I. The theorems in section 2 will follow from the next lemma of Makarov. (See [6].)

Lemma 1.4 (Makarov). *If $b \in \mathcal{B}$ and I is an arc on $\partial\mathbb{D}$ then we have*

$$(*) \qquad \left| \frac{1}{|I|} \int_I (b(\zeta) - b(z_I))^n \, d\zeta \right| \leq Cn!\|b\|_{\mathcal{B}}^n \qquad n \geq 1.$$

Here, the integral is defined as the limit of the integrals of

$$(b(r\xi) - b(z_I))^n$$

as $r \to 1$.

Proof. We have

$$\int_0^1 \left(\log \frac{1+t}{1-t}\right)^n dt = 2 \int_{-\infty}^0 x^n \frac{e^x}{(1+e^x)^2} \, dx \leq 2 \int_{-\infty}^0 x^n e^x \, dx \leq 2n! \quad .$$

If $b(0) = 0$ and $I = [e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}] \subset \mathbb{T}$ then $(*)$ follows by twice integrating the inequality

$$|b'(z)| \leq \frac{\|b\|_{\mathcal{B}}}{1 - |z|^2}$$

on the imaginary axis from $-i$ to i and applying the Cauchy integral theorem. In general, let

$$g(z) = b(\tau(z)) - b(\tau(0))$$

where τ is the self mapping of $\mathbb{D}$ used to define the point z_I. We then have

$$\left| \int_I (b(\zeta) - b(z_I))^n \, d\zeta \right| = \left| \int_{-1}^1 (g(iy))^n \tau'(iy) \, dy \right|$$

and

$$(1 - |z|^2)|g'(z)| = (1 - |\zeta|^2)|b'(\zeta)| \qquad \zeta = \tau(z).$$

Now since
$$|\tau'(z)| = \frac{1-|z_I|^2}{|1+\overline{z_I}z|^2} \leq C|I|,$$
$(*)$ follows from the first case considered, and the proof is complete. □

We remark that if $b \in \mathcal{B}_0$ then we may replace the inequality
$$|b'(z)| \leq \frac{||b||_{\mathcal{B}}}{1-|z|^2}$$
in the above proof by
$$|b'(z)| \leq \frac{\beta(1-|z|)}{1-|z|}$$
for some function $\beta = \beta(\delta)$, determined by b, for which $\beta \to 0$ as $\delta \to 0$ and obtain the inequalities
$$\left| \frac{1}{|I|} \int_I (b(\zeta) - b(z_I))^n \, d\zeta \right| \leq Cn!\beta^n(|I|)||b||_{\mathcal{B}}^n \qquad n \geq 1.$$

For any $b \in \mathcal{B}$ let $b_r(z) = b(rz)$ for all $0 < r < 1$ and define
$$b_I = \lim_{r \to 1} \frac{1}{|I|} \int_I b_r(\zeta) |\, d\zeta|.$$
this limit exists by Cauchy's theorem because, as shown in the proof above, b is integrable on radii. Let $\mathcal{F}_n$ denote the increasing sequence of sigma algebras generated by the partition points $e^{2\pi i 2^{-k}}$, $1 \leq k \leq 2^n$ on the probability space $(\mathbb{T}, \frac{d\theta}{2\pi})$. Defining the $\mathcal{F}_n$ measurable function M_n by $M_n|I = b_I$ it is easily checked that $M = (M_n, \mathcal{F}_n)$ is a martingale. By lemma 1.4, and the definition of the Bloch space, we have for all n

1. $|M_n(\zeta) - b((1-2^{-n})\zeta)| \leq C||b||_{\mathcal{B}} \qquad \zeta \in \partial\mathbb{D}$
2. $|\Delta M_n(\zeta)| \leq C||b||_{\mathcal{B}} \qquad \zeta \in \partial\mathbb{D}$

We define a *Bloch martingale* to be a real dyadic martingale such that there exists $b \in \mathcal{B}$ with
$$\text{Re } b_I = S_n|I \qquad |I| = 2\pi 2^{-n}$$
for each dyadic interval I. The following lemma from [6], which we record without proof, characterizes Bloch martingales.

Lemma 1.5 (Makarov). *A dyadic martingale S is a Bloch martingale if and only if the differences $(S|I - S|J)$ are uniformly bounded for all adjacent pairs of dyadic intervals (I, J) with $|I| = |J|$.*

To make dimension estimates in the next section we will use

Lemma 1.6 (Hungerford)**.** *Fix $0 < \epsilon < c < 1$. Let $E_0 = \mathbb{T} = I_{0,0}$ and for $n > 1$, $E_n = \cup I_{n,k}$ where $I_{n,k}$ are disjoint closed arcs such that for each $I_{n,k}$ there is a unique $I_{n-1,j}$ with*

1. $I_{n,k} \subset I_{n-1,j}$
2. $|I_{n,k}| \leq \epsilon |I_{n-1,j}|$
3. $\sum_{i(j)} |I_{n,i}| \geq c|I_{n-1,j}|$, *where $i(j)$ runs through all indices such that $I_{n,i} \subset I_{n-1,j}$.*

Let $E = \bigcap_n E_n$. Then with dimE denoting the Hausdorff dimension of E, we have

$$dimE \geq 1 - \frac{\log c}{\log \epsilon}.$$

Proofs appear in [4] and [8].

2. Martingale proofs of some results of Rohde

The behavior of Bloch functions at the boundary of the disk is explained by the following lemma, which is a slight refinement of lemma[5.6] in [6].

Lemma 2.1. *Let $M = \{M_n\}$ be a complex dyadic martingale determined by a Bloch function b as explained in section 1. Assume that $M_0 = 0$ and that $|\Delta M_n| \leq 1$ for all n. Given $0 < \alpha < \frac{\pi}{2}$, there exist $0 < C_\alpha < 1$ and $a_0 > 0$ such that for all $a > a_0$ we have*

$$m\left(\left\{|arg\, M_\tau| < \frac{\pi}{2} - \alpha\right\}\right) > C_\alpha$$

where $\tau = \tau_a = \inf\{n : |M_n| \geq a\}$.

Proof. By familiar properties of the Fejer kernel, if we are given $\eta > 0$ we may choose $m(\eta)$ so that if $0 \leq \rho(t) \leq 1$ is nondecreasing and left continuous on $-\pi \leq t \leq \pi$ then

$$\sum_{\nu=-m}^{m} \frac{m+1-|\nu|}{m+1} \int_{-\pi}^{\pi} \rho(t) e^{-i\nu t}\, dt \geq 1 - \eta$$

$$\Longrightarrow \int_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}-\alpha} \rho(t)\, dt > C_\alpha > 0.$$

For each dyadic interval I we have by lemma 1.4,

$$(*) \qquad [(b-b_I)^n]_I \le C(n)n!\|b\|_{\mathcal{B}}^n$$

and

$$[(b-b_I)^n]_I = (b^n)_I + \sum_{k=1}^{n}(-1)^k\binom{n}{k}(b^{n-k})_I b_I^k$$

$$(**) \qquad = (b^n)_I - b_I^n + \sum_{k=1}^{n-2}(-1)^k\binom{n}{k}\left[(b^{n-k})_I - b_I^{n-k}\right] b_I^k$$

where the last equality follows from the identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

Letting M^n denote the martingale determined by b^n and with $0 \le n \le m(\alpha)$, we have

$$\int M_\tau \, dm = 0$$

and

$$\begin{aligned} \int M_\tau^n \, dm &= -\int [(M-M_\tau)^n]_\tau \; dm \\ &+ \int \sum_{k=1}^{n-2}(-1)^k\binom{n}{k}\left[(M^{n-k})_\tau - M_\tau^{n-k}\right] M_\tau^k \, dm. \end{aligned}$$

Applying $(**)$ and $(*)$ to the terms in square brackets, we have by induction that

$$\left|\int M_\tau^n \, dm\right| \le C(m) + C'(m)a^{n-2} \qquad 0 \le n \le m(\alpha).$$

Let $\rho(t), \quad -\pi \le t \le \pi$ be the distribution density of $\arg M_\tau$. From the above argument we have

$$\left|\int_{-\pi}^{\pi} e^{int}\rho(t)\, dt\right| \le C(m)a^{-2} \qquad 0 \le n \le m(\alpha)$$

so that

$$\sum_{\nu=-m}^{m}\frac{m+1-|\nu|}{m+1}\int_{-\pi}^{\pi}\rho(t)e^{-i\nu t}\, dt \ge 1 - C(m)a^{-2} \ge 1-\eta$$

if a is large enough. By the first paragraph of the proof this implies that

$$m\left(\left\{|\arg M_\tau| < \frac{\pi}{2} - \alpha\right\}\right) > C_\alpha$$

and completes the proof of the lemma. □

We now use lemma 2.1 to prove the theorems mentioned in the introduction.

Recall that there are singular inner functions S in the little Bloch space since there are singular measures with integrals in the little Zygmund class, ([7],[5],[11]). By a theorem of Frostman (see [3]), the functions

$$\frac{S-\lambda}{1-\bar{\lambda}S} \qquad \lambda \in \mathbb{D}$$

are non-finite Blaschke products except for a set of $\lambda \in \mathbb{D}$ with logarithmic capacity zero. Hungerford showed in [4] that the zeros of such products must always accumulate on a set with Hausdorff dimension 1. This result was strengthened by Rohde who proved the following theorem [10].

Theorem 2.2 (Rohde). *An inner function in the little Bloch space which is not a finite Blaschke product has, for each $\delta \in \mathbb{D}$, the non-tangential limit δ on a set $E_\delta \subset \mathbb{T}$ with Hausdorff dimension 1.*

Proof. Fix $\delta \in \mathbb{D}$ and choose N such that $2^{-N} < \frac{1}{2}(1-|\delta|)$. Let D_n denote the disk of radius $2^{-(N+n)}$ centered at δ. Since b is not a finite Blaschke product, given $\eta > 0$ we can find a dyadic arc I with $|I| < \eta$, $|b_I - \delta| < \eta$ and $(1-|z|^2)|b'(z)| < \eta$ for all z with $|z| > 1 - \frac{|I|}{2\pi}$. Taking $\eta > 0$ sufficiently small and applying lemma 2.1 to the function $\frac{b-b_I}{\eta}$, we can find a dyadic interval $E_0 = I^0$ with $b_{I^0} \in D_2$ and $|I^0| < \eta$. Because $|b| \to 1$ nontangentially almost everywhere, the set of maximal dyadic subintervals $\{J_j\}$ with $b_{J_j} \notin D_0$ has

$$\sum_j |J_j| = |I^0|$$

and given $\epsilon > 0$, if $\eta > 0$ is small enough then since η controls $(1-|z|^2)|b'(z)|$ which in turn controls ΔM_n, we have $|J_j| < \epsilon |I^0|$ for all j. We apply lemma 2.1 in each J_j to obtain $0 < C < 1$ and a set of dyadic intervals $E_1 = \{I_k^1\}$ with $b_{I_k^1} \in D_2$ for all k and

$$\sum_k |I_k^1| \geq C|I^0|.$$

We continue the construction in the obvious way, obtaining $E_0, E_1, \ldots, E_{N_1}$. Because $(1-|z|^2)|b'(z)| \to 0$ as $|z| \to 1$, with N_1 sufficiently large we may alter the construction by switching to the disks (D_1, D_3) in place of (D_0, D_2) and keep the same numbers $\epsilon > 0$ and $C > 0$. In general, for $n > 1$, when N_n is sufficiently large we may change the construction by switching to the disks (D_n, D_{n+2}) in place of (D_{n-1}, D_{n+1}), keeping the same $\epsilon, C > 0$. By lemma 1.6 the set $E = \bigcap_k E_{N_k}$ has

$$\dim E \geq 1 - \frac{\log C}{\log \epsilon}$$

and by lemma 1.4 and the remark following it, $b \to \delta$ non-tangentially at each point of E. Since $\epsilon > 0$ may be chosen arbitrarily small in the above argument, the theorem is proved. □

Theorem 2.3 (Rohde). *Let $\{Q_n\}$ be a sequence of squares in the plane with pairwise disjoint interiors, edges parallel to the coordinate axes and of length $a > 0$, and such that Q_n is adjacent to Q_{n+1} for all n. There is a universal constant $K > 0$ such that if $b \in \mathcal{B}$ has nontangential limits almost nowhere then there exists a set E with*

$$dim\ E \geq 1 - a^{-1} K \|g\|_{\mathcal{B}}$$

such that, for each $\zeta \in E$ we can find $r_n \to 1$ with

$$b(r\zeta) \subset Q_n \cup Q_{n+1} \qquad \text{for } r_n \leq r \leq r_{n+1}.$$

PROOF. We may assume that $\|b\|_{\mathcal{B}} \leq 1$. The theorem is then interesting for large values of a. Let Q'_n denote the square of edge length $\frac{a}{2}$ concentric with Q_n and with parallel edges. Let D_n denote the disk of radius $\frac{a}{20}$ concentric with the square Q_n. Consider any two adjacent squares Q_n and Q_{n+1}. Let R_n denote the smallest rectangle containing $Q'_n \cup Q'_{n+1}$. For sufficiently large $a > 0$ independent of $b \in \mathcal{B}$, if $b_I \in D_n$ then by the assumption on the function b, finitely many applications of lemma 2.1 show that there exists a universal constant $0 < C < 1$ and a collection $\{I_j\}$ of dyadic subintervals of I with $\sum |I_j| \geq C|I|$, such that for each j we have $B_{I_j} \in D_{n+1}$ and $b_J \in R_n$ for all dyadic J with $I_j \subset J \subset I$. Because $\|b\|_{\mathcal{B}} \leq 1$ we have $|I_j| \leq c2^{-[a]}|I|$ for all j. Again, by the assumption on b, we find after finitely many applications of lemma 2.1, a dyadic interval with $b_{I_0} \in D_0$. Constructing the appropriate nested sequence of intervals contained in I_0, the proof is completed by applying lemma 1.6 and lemma 1.4 with $n = 1$. □

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