Math 32 10-9-09.

Thm: Suppose that $\vec{F} = (f, g)$ is a smooth vector field defined on

$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

s.t.
$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{in } R.$$

Then there is a smooth function $\varphi(x,y)$ defined on R s.t.
$$\vec{\nabla}\varphi = \vec{F}.$$

Pf. We want to solve (for φ)

$$\frac{\partial \varphi}{\partial x} = f(x,y)$$

$$\frac{\partial \varphi}{\partial y}(x,y) = g(x,y).$$

Choose an point (x_0, y_0) in R. and integrate the 1st equation in the 1st variable

i.e. ~~let~~

$$\cancel{\varphi(x,y) = \int_{x_0}^{x} f(t,y)}$$

$$\int_{x_0}^{x} \frac{\partial \varphi}{\partial x}(t,y)\,dt = \int_{x_0}^{x} f(t,y)\,dt.$$

Thm: Suppose that $\vec{F} = (f, g)$ is a smooth vector field defined on

$$R = \{(x,y) : a \le x \le b, \ c \le y \le d\}.$$

s.t $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ in R.

Then there is a smooth function $\varphi(x,y)$ defined on R s.t.

$$\nabla \vec{\varphi} = \vec{F}$$

Pf: We want to solve the system

$$\frac{\partial \varphi}{\partial x} = f(x,y)$$

$$\frac{\partial \varphi}{\partial y} = g(x,y)$$

for the function φ.

Integrate the 1st equation.
Choose a point (x_0, y_0) in R and let

$$\varphi(x,y) = \int_{x_0}^{x} f(t,y)\,dt + u(y)$$

where $u(y)$ is an (for now) arbitrary function of y.

then

$$\frac{\partial \varphi}{\partial x} = f(x,y) + \frac{\partial}{\partial x} u(y) = f(x,y)$$

so the 1st equation is satisfied.

Can we also satisfy the 2nd equation by choosing u carefully?.

$$\frac{\partial}{\partial y}\left[\int_{x_0}^{x} f(t,y)\, dt + u(y) \right]$$

$$= \int_{x_0}^{x} \frac{\partial}{\partial y} f(t,y)\, dt + u'(y)$$

$$= \int_{x_0}^{x} \frac{\partial}{\partial x} g(t,y)\, dt + u'(y)$$

$$= g(x,y) - g(x_0,y) + u'(y).$$

So the answer is yes if we let

$$u'(y) = g(x_0,y).$$

i.e $$u(y) = \int_{y_0}^{y} g(x_0, t)\, dt.$$

We now see that

$$\varphi(x,y) = \int_{x_0}^{x} f(t,y)\, dt + \int_{y_0}^{y} g(x_0, t)\, dt$$

is a potential function, and this completes the proof.

Now suppose

$$\vec{F} = \langle f_1, f_2, f_3 \rangle \quad \text{in}$$

$$R = \{(x,y,z) : x_1 \le x \le x_2,\ y_1 \le y \le y_2,\ z_1 \le z \le z_2\}.$$

and $D_1 f_2 = D_2 f_1$, $D_1 f_3 = D_3 f_1$, $D_2 f_3 = D_3 f_2$.

Try to solve.

$$\frac{\partial \varphi}{\partial x}(x,y,z) = f_1(x,y,z)$$

$$\frac{\partial \varphi}{\partial y}(x,y,z) = f_2(x,y,z)$$

$$\frac{\partial \varphi}{\partial z}(x,y,z) = f_3(x,y,z).$$

let

$$\varphi(x,y,z) = \int_{x_0}^{x} f_1(t,y,z)\,dt + u(y,z).$$

so that $\dfrac{\partial\varphi}{\partial x} = f_1(x,y,z)$.

then

$$\frac{\partial}{\partial y}\left[\int_{x_0}^{x} f_1(t,y,z)\,dt + u(y,z)\right]$$

$$= \int_{x_0}^{x} \frac{\partial f_1}{\partial y}(t,y,z)\,dt + \frac{\partial u}{\partial y}(y,z)\Big]$$

$$= \int_{x_0}^{x} \frac{\partial f_2}{\partial x}(t,y,z)\,dt + \frac{\partial u}{\partial y}(y,z)$$

$$= f_2(x,y,z) - f_2(x_0,y,z) + \frac{\partial u}{\partial y}(y,z).$$

so the 2nd equation will be satisfied if $\dfrac{\partial u}{\partial y} = f_2(x_0,y,z)$.

let $u(y,z) = \displaystyle\int_{y_0}^{y} f_2(x_0,t,z)\,dt + v(z)$.

then

$$\frac{\partial \varphi}{\partial y} = f_2(x_0, y, z)$$

and

$$\varphi(x, y, z) = \int_{x_0}^{x} f_1(t, y, z)\, dt + \int_{y_0}^{y} f_2(x_0, t, z)\, dt + V(z).$$

to satisfy the 3rd equation we need.

$$\frac{\partial}{\partial z}\left[\int_{x_0}^{x} f_1(t, y, z)\, dt + \int_{y_0}^{y} f_2(x_0, t, z)\, dt + V(z)\right]$$

$$= \int_{x_0}^{x} \frac{\partial f_3}{\partial x}(t, y, z)\, dt + \int_{y_0}^{y} \frac{\partial f_3}{\partial y}(x_0, t, z)\, dt + V'(z)$$

$$= f_3(x, y, z) - f_3(x_0, y, z) + f_3(x_0, y, z) - f_3(x_0, y_0, z) + V'(z)$$

(to be...)

$$= f_3(x, y, z)$$

So we need.

$$f_3(x_0, y_0, z) = V'(z).$$

take $$v(z) = \int_{z_0}^{z} f_3(x_0, y_0, t)\, dt.$$

then $v'(z) = f_3(x_0, y_0, z)$

and we have.

$$\varphi(x,y,z) = \int_{x_0}^{x} f_1(t,y,z)\, dt + \int_{y_0}^{y} f_2(x_0,t,z)\, dt + \int_{z_0}^{z} f_3(x_0,y_0,t)\, dt$$

Curve Integrals (AKA Line Integrals, Path Integrals).

Def: let $\vec{F}$ be a smooth vector field.
$\vec{C}(t) = C$ a smooth curve.
$(a \leq t \leq b)$.

then
$$\int_C \vec{F} \equiv \int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t)\, dt.$$

For any regular parametrization (i.e. $\vec{c}\,'(t) \neq 0 \quad a \le t \le b$).

we have

$$\int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt.$$

$$= \int_0^L \vec{F}(\vec{c}(t(s))) \cdot \vec{c}\,'(t(s))\, t'(s)\, ds.$$

where s denotes arc length on the curve and L is the total length.

and this is.

$$\int_0^L \vec{F}(\vec{c}(s)) \cdot \vec{c}\,'(s)\, ds.$$

with $\vec{c}$ parametrized by arc length.

So the integral doesn't depend on the (regular) parametrization.

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$$\vec{F}(x,y) = (x^2 - 2xy,\ y^2 - 2xy)$$

$\vec{C}$: parabola $y = x^2$ from $(-2,4)$ to $(1,1)$.

$$\vec{c}(t) = \langle t, t^2 \rangle \qquad -2 \le t \le 1.$$

$$\int_{\vec{C}} \vec{F} = \int_{-2}^{1} \langle t^2 - 2t^3,\ t^4 - 2t^3 \rangle \cdot \langle 1, 2t \rangle \cdot dt$$

$$= \int_{-2}^{1} \left((t^2 - 2t^3) + 2t^5 - 4t^4 \right) dt$$

$$= \frac{1}{3}t^3 - \frac{1}{2}t^4 + \frac{2}{6}t^6 - \frac{4}{5}t^5 \Big|_{-2}^{1}$$

$$= \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{4}{5} \right) - \left(-\frac{8}{3} - \frac{16}{2} + \frac{128}{6} + \frac{128}{5} \right).$$

$$= \frac{10}{3} + \frac{15}{2} - \frac{4}{5} - \frac{128}{6} - \frac{128}{5}$$

$$= \frac{10}{3} + \frac{15}{2} - \frac{128}{6} - \frac{132}{5}. \ \ldots$$

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Work

A constant force F applied to an object through a distance d performs <u>Work</u>

$$W = Fd.$$

<u>e.g.</u> lifting a 120 lb bag of sand 5 ft takes 600 ft-lb of work.

The work done by a variable Force $F(x)$ in moving an object along the x-axis from $x=a$ to $x=b$ is

$$\int_a^b F(x)\,dx.$$

We approximate the force as $F(x_i^*)$ on an interval (x_i, x_{i+1}) with

$x_i \le x_i^* \le x_{i+1}$ and $(\Delta x)_i = x_{i+1} - x_i$ small.

$a = x_0 < x_1 < \dots < x_n = b.$

The work done in moving the object from x_i to x_{i+1} is approximately

$$F(x_i^*)(\Delta x)_i$$

and summing this over all the i's gives

$$\sum_{i=0}^{n-1} F(x_i^*)(\Delta x)_i$$

which is a Riemann - Sum for

$$\int_a^b F(x)\,dx.$$

If the Force F remains constant but does not point in the direction of the (straight line) path of motion, we take $F = \vec{F}$.

take the motion to be from $\vec{P}$ to $\vec{Q}$.

and decompose $\vec{F}$ as.

$$\vec{F} = \frac{\vec{F} \cdot \vec{PQ}}{\|\vec{PQ}\|^2}\,\vec{PQ} + \left(\vec{F} - \frac{\vec{F}\cdot\vec{PQ}}{\|\vec{PQ}\|^2}.\vec{PQ}\right)$$

The second part is perpendicular to the path of motion and performs <u>NO</u> <u>WORK</u>

The 1st part acts ~~in th~~ parallel to the motion and we may take the work to be.

$$\frac{\vec{F}\cdot\vec{PQ}}{\|\vec{PQ}\|}\,\|\vec{PQ}\| = (\vec{F}\cdot\vec{PQ})$$

Notice that this is positive if the projection of $\vec{F}$ onto $\vec{PQ}$ points in the <u>same</u> direction as $\vec{PQ}$ and it is negative if the projection is in the opposite direction.

To summarize:

The work W done by a constant force $\vec{F}$ on an object moving from $\vec{P}$ to $\vec{Q}$ is

$$W = \vec{F} \cdot \vec{PQ} \equiv \vec{F} \cdot (\vec{Q} - \vec{P}).$$

<u>e.g.</u> The work done by the constant force $\vec{F} = 2\vec{i} + 3\vec{j} + \vec{k}$ when it moves a particle along the line from $P(1,0,-1)$ to $Q(3,1,2)$ is:

$$\langle 2,3,1 \rangle \cdot \langle 2,1,3 \rangle = 10$$

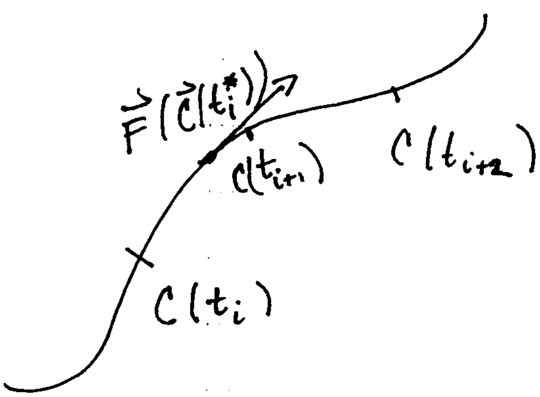
Now consider an object moving along a curve $\vec{C} = \vec{C}(t)$ $a \le t \le b$. in a variable force field $\vec{F} = \vec{F}(x, y, z)$.
(in $\mathbb{R}^2$ or $\mathbb{R}^3$).

If $\vec{C}(t) = \langle x(t), y(t), z(t) \rangle$, the force on the "particle" at time t is

$$\vec{F}(x(t), y(t), z(t)).$$

Let $a = t_0 < t_1 < t_2 < \dots < t_n = b$

with $t_{i+1} - t_i = (\Delta t)_i$ small for each i.



When $(\Delta t)_i$ is small

$$\vec{C}(t_{i+1}) - \vec{C}(t_i) \approx (x'(t_i^*), y'(t_i^*), z'(t_i^*)) \cdot (\Delta t_i)$$

for some t_i^* with $t_i \le t_i^* \le t_{i+1}$.

and

$$\vec{F}(\vec{C}(t_i^*)) \cdot (x'(t_i^*), y'(t_i^*), z'(t_i^*))(\Delta t)_i$$

approximates the work done by the Force field $\vec{F}$ on the particle when it moves from $\vec{C}(t_i)$ to $\vec{C}(t_{i+1})$ along the curve.

Adding these approximations over all i gives

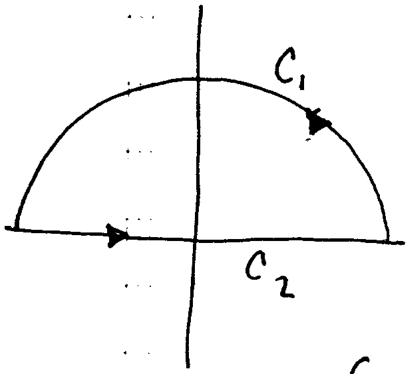
$$\sum_{i=0}^{n-1} \vec{F}(\vec{C}(t_i^*)) \cdot (x'(t_i^*), y'(t_i^*), z'(t_i^*))(\Delta t)_i$$

which is a Riemann sum for.

$$\int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t)\, dt. \equiv \int_{\vec{C}} \vec{F}$$

(the $\overset{\text{curve path}}{\text{line}}$ integral of $\vec{F}$ on the path $\vec{C}$).

e.g. Find the work done by the force field $\vec{F} = (x^2+y^2)\vec{i} + (x+y)\vec{j}$ as an object moves counterclockwise along the circle $x^2+y^2=1$ from $(10,0)$ to $(-1,0)$ and then back to $(1,0)$ along the x-axis.



$\vec{C}_1 : \vec{C}_1(t) = \langle \cos t, \sin t \rangle, \ 0 \le t \le \pi$

$\vec{C}_2 : C_2(t) = \langle t, 0 \rangle \quad -1 \le t \le 1.$

$\vec{F}(C_1(t)) \cdot C_1'(t) = \langle 1, \cos t + \sin t \rangle \cdot \langle -\sin t, \cos t \rangle$

$$\int_{C_1} \vec{F} = \int_0^{\pi} (-\sin t + \cos^2 t + \sin t \cos t)\, dt.$$

$$-\int_0^{\pi} \sin t \, dt = \cos(\pi) - \cos(0) = -2$$

$$\int_0^{\pi} \cos^2 t \, dt = \frac{1}{2} \int_0^{\pi} (\cos^2 t + \sin^2 t)\, dt = \pi/2.$$

$$\int_0^{\pi} \sin t \cos t \, dt. = \frac{1}{2} \int_0^{\pi} \sin 2t \, dt = 0.$$

so $\displaystyle\int_{C_1} \vec{F} = \pi/2 - 2.$

$\vec{F}(C_2(t)) \cdot (C_2'(t)) = \langle t^2, t \rangle \cdot \langle 1, 0 \rangle = t^2$

$$\int_{C_2} \vec{F} = \int_{-1}^{1} t^2 \, dt = \frac{1}{3} t^3 \Big|_{-1}^{1} = 2/3.$$

So $W = \frac{\pi}{2} + \frac{2}{3} - 2 = \frac{\pi}{2} - \frac{4}{3}.$

We saw last time that the value of

$$\int_C \vec{F}$$ does not depend on the

parametrization used to describe $\vec{C}$.

When $\vec{F} = \langle u(x,y,z), v(x,y,z), w(x,y,z) \rangle$

it is customary to write

$$\int_C \vec{F} \quad \text{as} \quad \int_C u\,dx + v\,dy + w\,dz$$

$$\text{or} \quad \oint_C u\,dx + v\,dy + w\,dz.$$

the definition is as before (i.e)

$$= \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$$

where $\vec{c}(t)$ is <u>any</u> convenient parametrization of the ~~$\vec{a}$~~ the curve C.

e.g. $\int_C y\,dx - x\,dy + dz$

C is the helical path

$x = 3\sin t \quad y = 3\cos t \quad z = t \quad 0 \le t \le \pi/2.$

The integral is

$$\int_0^{\pi/2} (3\cos t \cdot 3\cos t - 3\sin t(-3\sin t) + 1)\,dt.$$

$$= \int_0^{\pi/2} 10\,dt = 5\pi.$$

e.g. $$\int_C \frac{x\,dy - y\,dx}{x^2+y^2}$$

C is the unit circle traversed once counter clockwise.

(This is $\int_C \vec{F}$ with $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.)

let $\vec{C}$: $\vec{C}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

then

$$\int_C \vec{F} = \int_0^{2\pi} -\sin t(-\sin t) + \cos t \cos t\,dt = 2\pi.$$