Boundary behavior of harmonic functions and Brownian motion.

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Outline

1. Harmonic Measure
   - Harmonic measure and Brownian motion
   - McMillan's twist point theorem
   - Dimension estimates.
2. Probabilistic Approaches
   - Twist point theorem
   - Fatou's theorem
3. Extensions to NTA domains
   - A Fatou theorem
   - A twist point theorem
   - Dimension estimates?

Outline

Harmonic Measure and Brownian Motion

- Harmonic measure is the exit distribution of Brownian motion started from a given point z_0 in a given domain.
- For domains with smooth boundary, it is also $\frac{\partial g}{\partial n}|ds|$ where g is Green's function with pole at z_0.

Outline

McMillan's twist point theorem

For simply connected domains in the plain, and with respect to the harmonic measure, almost every boundary point is either a twist point or a cone point.

Cone Points

A cone point in a planar domain is any boundary point which is the vertex of a triangle contained in the interior of the domain.

Twist Points

A twist point is a boundary point ξ of a planar domain such that

$$-\liminf \arg\,(z-\xi) = \limsup \arg\,(z-\xi) = +\infty$$

where the limes are taken as z approaches ξ along any curve contained in the interior of the domain and arg denotes a fixed single valued branch of the argument.

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In two dimensions

- For any planar domain, the harmonic measure is supported on a set of σ- finite one dimensional Hausdorff measure. (Makarov, Jones-Wolff, Wolff)
- In the infinitely connected case, the dimension can be less than 1. (Carleson)

In two dimensions

For simply connected domains:

- The restriction of the harmonic measure to the cone points is in the same measure class as the one dimensional Hausdorff measure λ_1.
- On the set of twist points, the harmonic measure and λ_1 are mutually singular

In three or more dimensions

For domains in $\mathbb{R}^n$ with $n \geq 3$:

- There is $\tau(n) > 0$ such that the dimension of the support of the harmonic measure is always less than $n - \tau(n)$. (Bourgain)
- There is a domain Ω in $\mathbb{R}^3$ and there is $\epsilon > 0$ such that no subset of $\partial\Omega$ with Hausdorff dimension less than $2 + \epsilon$ can have full harmonic measure.

Outline

Using Ito's formula

- A single valued, continuous and harmonic branch of the argument of the gradient of Green's function can be defined in the domain (minus a slit, or one may condition and then think locally).
- $u(X_t)$ is a martingale and a time change of a Brownian motion, perhaps stopped at a stopping time.

Using Ito's formula

- e.g. $\liminf u(X_t) < +\infty$ a.s., (in fact, LIL for this martingale and its conjugate can be used to derive the Makarov dimension estimates.)
- geometric arguments connect the almost sure exit behavior of Brownian motion to the behavior of the trajectories of the Green function

Lipschitz Subdomains and Stopping Times

To finish (one half of) the proof of the twist point theorem, one must show that the set of boundary points which are not cone points but for which $\limsup U(X_t) < +\infty$ has harmonic measure zero.

There are two main ingredients:

- Fatou's theorem for convergence of bounded harmonic functions along Green lines.
- Construction of Lipschitz sub-domains and use of stopping time arguments with the strong Markov property.

Outline

Why reprove Fatou's theorem?

- To try to make arguments in higher dimesnions relating exit of Brownian motion to the behavior at the boundary of the Green lines, one will need the same sort of Fatou theorem along the Green trajectories.
- The same question is posed by R. Bass in the book Probabilistic Techniques in Analysis (chapter 3, pg. 213)

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Definition of NTA domains

A Non Tangentially Accessible (NTA) domain is one in which:

- each boundary point has both an interior and exterior non-tangential ball at every scale and
- non-tangential balls of the same scale for different boundary points can be connected by a chain of non-tangential balls whose length is comparable to the Euclidean distance between the balls.

Fatou on Green lines

- Adjustments to the reasoning for the two dimensional proof give a Fatou theorem for Green lines in NTA domains.
- The main ingredients are again the Strong Markov property and the construction of stopping time subdomains.

Outline

A dimension 3 analog for NTA domains

Let η be a Brownian path and let $\tau(\eta)$ denote its first exit time from a given NTA domain Ω.

Let

$$E(\eta) = \bigcap_{\epsilon>0} \overline{\{\frac{X_t(\eta) - X_\tau(\eta)}{|X_t(\eta) - X_\tau(\eta)|} : \tau - \epsilon < t < \tau\}}$$

Then $E(\eta)$ is almost surely either a half sphere or a sphere, and the half sphere exits are almost surely at cone points.

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Cone points

- A theorem of Dahlberg shows that in Lipshitz domains, harmonic measure and surface measure are in the same measure class.
- It is not hard to see that harmonic measure restricted to cone points is in the same measure class as the (n-1) dimensional Hausdorff measure.

Twist points

So the interesting part of the harmonic measure is again at the twist points.

- If one can find an explicit geometric charecterization of the twist points which support harmoic measure in, say, the Von koch snowflake perhaps that would help to characterize the corresponding points in the Wolff snowball example.
- What more can be said about the process $\frac{\nabla g(X_t)}{|\nabla g(X_t)|}$?