MATH 149, FALL 2008, TAKE-HOME MIDTERM

Please print your name clearly!

Name:____

This test is due on Wednesday, 11/5/08, in class. While completing it, feel free to use your lecture notes from class, as well as those posted on the class webpage. You are however not allowed to consult with anyone: it is understood that solutions to this midterm represent solely your own work with no outside assistance. Good luck!

Problem 1. (10 points) Let A and B be two sets in $\mathbb{R}^N$, $N \geq 1$. Minkowski sum of A and B is defined by

$$A + B = \{\boldsymbol{a} + \boldsymbol{b} : \boldsymbol{a} \in A,\ \boldsymbol{b} \in B\}.$$

Suppose that A and B are compact convex sets. Prove that $A + B$ is also a compact convex set.

Problem 2. (20 points) Let Λ be a lattice of rank r in $\mathbb{R}^N$, $1 \leq r < N$, and let $V = \operatorname{span}_{\mathbb{R}} \Lambda$.

a) - (10 points) Prove that Λ is a discrete co-compact subgroup of V.

b) - (5 points) Is Λ a discrete co-compact subgroup of $\mathbb{R}^N$? If your answer is "yes", give a proof; if your answer is "no", explain which properties fail and why.

c) - (5 points) Is V a discrete co-compact subgroup of $\mathbb{R}^N$? If your answer is "yes", give a proof; if your answer is "no", explain which properties fail and why.

Problem 3. (10 points) Define sets

$$A = \{(x, x^2) : x \in \mathbb{R}\},$$
$$B = \{(x, x^3) : x \in \mathbb{R}\}.$$

Are A and B homeomorphic? Diffeomorphic? Prove your answers.

Problem 4. (20 points) Let a, b be positive real numbers, and suppose that

$$\Lambda = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 1 & a & b \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^4$$

is a full-rank sublattice of $\mathbb{Z}^4$.

a) - (5 points) What can you say about a and b? Prove your answer.

b) - (10 points) Suppose that $a = b$ and Ω is a full-rank sublattice of Λ, such that the volume of a fundamental domain of Ω in $\mathbb{R}^4$ is equal to 20. What can you say about a? Prove your answer.

c) - (5 points) Assuming part b, can Λ be a sublattice of any of the following two lattices:

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix} \mathbb{Z}^4, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 3 \end{pmatrix} \mathbb{Z}^4,$$

and if so, which one(s)? What can you say about a in each case? Prove your answers.

Problem 5. (10 points) Let $N \geq 2$ be even, and define quadratic forms

$$Q_1(X_1, \ldots, X_N) = \sum_{i=1}^{N} X_i^2 - X_1X_2 - X_3X_4 - \cdots - X_{N-1}X_N,$$

and

$$Q_2(X_1, \ldots, X_N) = \boldsymbol{X}^t B \boldsymbol{X}$$

for some real symmetric matrix B. Suppose that Q_1 and Q_2 are isometric.

a) - (5 points) What can you say about the eigenvalues of B? Prove your answer.

b) - (5 points) Let

$$Q_1'(X_1, \ldots, X_N) = \frac{1}{2}\sum_{i=1}^{N} X_i^2 - X_1X_2 - X_3X_4 - \cdots - X_{N-1}X_N.$$

Are Q_1' and Q_2 isometric? Prove your answer.

Problem 6. (10 points) Let $N \geq 2$, and let

$$X = \{\boldsymbol{x} \in \mathbb{R}^N : F(\boldsymbol{x}) \leq 1\}$$

be a $\mathbf{0}$-symmetric star body whose distance function F satisfies the triangle inequality:

$$F(\boldsymbol{x} + \boldsymbol{y}) \leq F(\boldsymbol{x}) + F(\boldsymbol{y}),$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. Let Λ be a lattice of full rank in $\mathbb{R}^N$. Prove that for every real number

$$\mu \geq 2 \left(\frac{\det(\Lambda)}{\mathrm{Vol}(X)} \right)^{1/N}$$

the intersection $\mu X \cap \Lambda$ contains a non-zero vector. Is this statement still true if F does not satisfy the triangle inequality? Either prove your answer or give a counter-example.

Problem 7. (10 points) Let $Q(\boldsymbol{X})$ be a quadratic form in N variables with real coefficients. Let H^+ be the set of all autometries σ of Q such that $\det(\sigma) > 0$, and let H^- be the set of all autometries τ of Q such that $\det(\tau) < 0$. Are H^+ and H^- subgroups of $\mathrm{GL}_N(\mathbb{R})$? Prove your answer.

Problem 8. (10 points) Let Λ be a lattice of full rank in $\mathbb{R}^N$, and let $M \subset \mathbb{R}^N$ be a compact Jordan measurable set such that $\mathrm{Vol}(M) < \det(\Lambda)$. Prove that there exists a point $\boldsymbol{x} \in \mathbb{R}^N$ such that the intersection $M \cap (\Lambda + \boldsymbol{x})$ is empty.