MATH 60, FALL 2008, TEST I

Please print your name clearly!

Name: SOLUTIONS

Please show all your work, that is explain every step of your solution - it is your work, not just the answer, that is being evaluated. When asked to prove a statement, make sure to provide reasoning behind each claim you are making in the process of the proof. The use of calculators or any other electronic devices is prohibited during the test. You are also not allowed to use any study materials except for those provided to you during the test. Cheating is strictly prohibited, and will be prosecuted. Good luck!

Problem 1. (40 points) Let λ be a real number, and let

$$a_1 = \begin{pmatrix} \lambda \\ 0 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

a) (20 points) For which values of λ are these three vectors linearly dependent? Prove your answer.

b) (20 points) How many solutions does the linear system

$$\cancel{\left.\begin{aligned} x_1 + x_3 &= 1 \\ x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_3 &= 3 \end{aligned}\right\}} \qquad \left.\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ - x_2 &= 2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}\right\}$$

have? Prove your answer. Find these solutions. Show all work.

a) Let $A = (\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3) = \begin{pmatrix} \lambda & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$,

then $\det A = -\lambda + 2$, so $\det A \neq 0$ whenever $\lambda \neq 2$

Thus $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are linearly dependent iff $\lambda = 2$.

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b) The coefficient matrix of this linear system is

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

which is A from part (a) above with $\lambda = 1$ $\Rightarrow$

$\Rightarrow$ by part (a), B is non-singular $\Rightarrow$

$\Rightarrow$ this system has a unique solution, given by $\vec{x} = B^{-1} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Now we find B^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\mathrm{III} \leftrightarrow \mathrm{III} + \mathrm{II}]{\mathrm{I} \leftrightarrow \mathrm{I} + \mathrm{II}}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow[\substack{\mathrm{II} \leftrightarrow (-1) \cdot \mathrm{II} \\ \mathrm{III} \leftrightarrow (-1) \cdot \mathrm{III}}]{\mathrm{III} \leftrightarrow \mathrm{III} - \mathrm{I}}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \xrightarrow{\mathrm{I} \leftrightarrow \mathrm{I} - 2 \cdot \mathrm{III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

Therefore: $B^{-1} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$,

and so the only solution to our linear system is:

$$\vec{x} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -2 \end{pmatrix}.$$

Problem 2. (35 points) Suppose that U and W are subspaces of a vector space V such that U is not contained in W and W is not contained in U. Define the set

$$S = \{v \in V : v \in U \text{ or } v \in W\}.$$

In other words, S is the union of U and W. Is S a subspace of V? Prove your answer.

S is <u>not</u> a subspace of V.

<u>Proof</u>: Let $\vec{x} \in U$ such that $\vec{x} \notin W$, and let $\vec{y} \in W$ such that $\vec{y} \notin U$. Then

$$\vec{x}, \vec{y} \in S.$$

Suppose $\vec{x}+\vec{y} \in S \Rightarrow \vec{x}+\vec{y} \in U$ OR $\vec{x}+\vec{y} \in W$. If $\vec{x}+\vec{y} \in U$, then:

$$-\vec{x} + (\vec{x}+\vec{y}) = \vec{y} \in U \text{ - contradiction.}$$

If $\vec{x}+\vec{y} \in W$, then:

$$-\vec{y} + (\vec{x}+\vec{y}) = \vec{x} \in W \text{ - contradiction.}$$

Therefore $\vec{x}+\vec{y} \notin S \Rightarrow S$ is not a subspace of V, by the subspace criterion. ✓

Problem 3. (25 points) Let

$$f(x) = \frac{1}{x-1},\ g(x) = x+1,\ h(x) = \frac{x^2}{x-1}$$

be functions in the vector space $C^\infty[2,3]$, which is the space of all continuous functions with infinitely many continuous derivatives on the interval [2,3]. Let

$$V = \text{span}\{f(x), g(x), h(x)\},$$

so V is a subspace of $C^\infty[2,3]$. What is the dimension of V? Prove your answer.

Notice that

$$h(x) = \frac{x^2}{x-1} = \frac{1}{x-1} + x + 1 = f(x) + g(x),$$

therefore:

$$\text{span}\{f(x), g(x), h(x)\} = \text{span}\{f(x), g(x)\}$$

Now compute the Wronskian of $f(x)$ and $g(x)$:

$$W(f,g) = \det\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} =$$

$$= \det\begin{pmatrix} \frac{1}{x-1} & x+1 \\ -\frac{1}{(x-1)^2} & 1 \end{pmatrix} = \frac{1}{x-1} + \frac{x+1}{(x-1)^2} =$$

$$= \frac{2x}{(x-1)^2} \neq 0 \text{ on } [2,3] \Rightarrow f(x) \text{ and } g(x)$$ are linearly independent.

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Therefore

$$V = \text{span}\{f(x), g(x)\},$$

and $f(x), g(x)$ form a basis for $V \Rightarrow$

$\Rightarrow$ dimension of V is $\underline{\underline{2.}}$

Bonus question: (20 points) Let $m > 3$ be a positive integer, and let V be as in Problem 3 above. Let

$$W_m = \text{span}\left\{\frac{x^2}{x-1}, \frac{2x^2-1}{x-1}, \frac{3x^2-2}{x-1}, \ldots, \frac{mx^2-(m-1)}{x-1}\right\}.$$

For which values of m is W_m a subspace of V? A proper subspace of V? Prove your answer.

(**Hint:** Recall that if W is a subspace of V, we say that it is proper if $W \neq V$.)

Notice that

$$\frac{2x^2-1}{x-1} - \frac{x^2}{x-1} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} =$$

$$= x+1, \quad \text{on } [2,3].$$

Therefore $g(x) = x+1 \in W_m$ for every m.

Now

$$f(x) = \frac{1}{x-1} = \frac{x^2}{x-1} - (x+1),$$

so $f(x) \in W_m$ for every m. Therefore $V = \text{span}\{f(x), g(x)\}$ is contained in W_m for every m.

On the other hand:

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$$\frac{x^2}{x-1} = f(x) + g(x)$$

$$\frac{2x^2-1}{x-1} = f(x) + 2g(x)$$

$$\vdots$$

$$\frac{mx^2-(m-1)}{x-1} = f(x) + mg(x),$$

and so W_m is contained in V for every m. Hence <u>$V = W_m$ for every m.</u>

(In other words, W_m is a subspace of V for every m, but for no m is it a proper subspace of V).